MATH 418 COMPLEX VARIABLES
Homework 7 Solution

Due March 15, 2001

Note: If you have any questions about the solution, or you think there are some typos/errors in the solution, please e-mail me. I'll double-check it and then reply to you. Thank you.

C13. Proof: Suppose z is such that $|z| = \rho \in (|\beta|, |\gamma|)$, then we have

$$\sum_{-\infty}^{+\infty} |a_n z^n| = \sum_{0}^{+\infty} |a_n \gamma^n| \left|\frac{z^n}{\gamma^n}\right| + \sum_{-\infty}^{-1} |a_n \beta^n| \left|\frac{\beta^{-n}}{z^{-n}}\right| \leq \sum_{0}^{+\infty} |a_n \gamma^n| + \sum_{-\infty}^{-1} |a_n \beta^n| < \infty$$

So, the series above converges uniformly on the circle $|z| = \rho$.$\square$

C14.
(i) Proof: Since α is an isolated singularity of $f(z)$, there exists a neighborhood U of α, such that $f(z)$ is analytic in $U\backslash\{\alpha\}$. Then $g \circ f$ must be also analytic in $U\backslash\{\alpha\}$, for g is entire. Thus α has to be an isolated singularity of $g \circ f$.
(ii) If α is a removable singularity of f, then it must be also a removable singularity of $g \circ f$.
Proof: We note $\forall z \in U\backslash\{\alpha\}$, where U is defined as above, $g \circ f$ is differentiable at z. To see this, note $\lim_{\triangle z \longrightarrow 0} \frac{g(f(z+\triangle z))-g(f(z))}{\triangle z} = \lim_{\triangle z \longrightarrow 0} \frac{g(f(\triangle z+z))-g(f(z))}{f(\triangle z+z)-f(z)} \frac{f(\triangle z+z)-f(z)}{\triangle z}$. Since g is entire, it is differentiable at $f(z)$, so the first term in the RHS of the above equality has a limit. And since f is analytic in $U\backslash\{\alpha\}$, the second term in the RHS of the above equality has a limit, for it's just by definition the derivative of f at the point z. Hence, $\lim_{\triangle z \longrightarrow 0} \frac{g(f(z+\triangle z))-g(f(z))}{\triangle z}$ exists, i.e. $g \circ f$ is analytic at z. Since z is arbitrarily chosen, we conclude $g \circ f$ is analytic in $U\backslash\{\alpha\}$. And it's clear that if f can be continuously extended to α, then $g \circ f$ is also able to continuously extended to α. So, α is a removable singularity for $g \circ f$. $\square$

Problems in Levinson and Redheffer

2. Solution:
(i) $f(z) = \frac{1}{z(z-1)}$: The singularities are 0 and 1. $Res(f;0) = -1$, $Res(f;1) = 1$.
(ii) $f(z) = \frac{z}{z^4+1}$: The singularites are $e^{3\pi i/4}$, $e^{\pi i/4}$, $e^{-\pi i/4}$ and $e^{-3\pi i/4}$. They are all poles of order 1. We use formula (2.4) on page 191, and conclude the residues are $i/4, -i/4, i/4$, and $-i/4$, respectively.
(iii) $f(z) = \frac{\sin z}{z^2(\pi - z)}$: The singularities are 0 and π. $Res(f;0) = 1/\pi$, and $Res(f;\pi) = 0$.
(iv) $f(z) = \frac{ze^{iz}}{(z-\pi)^2}$: The singularity is π. The residue is $-\pi i - 1$.
(v) $f(z) = \frac{z^3+5}{(z^4-1)(z+1)}$: The singularities are 1, -1, i and $-i$. The residues are, respectively, $3/4$, $-9/4$, $(3+2i)/4$, and $(3-2i)/4$. $\square$

3. Proof: (i) In C, the integrand has two singularities: $i/2$ and $-i/2$. So, $\int_C f(z)dz = 2\pi i[Res(f;i/2)+Res(f;-i/2)] = 2\pi i[\frac{e^{\pi z}}{4(z+i/2)}|_{z=i/2} + \frac{e^{\pi z}}{4(z-i/2)}|_{z=-i/2}] = \pi i$. This result remains the same when C is the circle $\{|z| = 2\}$, since these two contours contain the same singularities. $\square$

(ii) The singularity of the integrand is 0. So, $\int_C \frac{e^z}{z^3} dz = 2\pi i \frac{d^2}{dz^2} e^z|_{z=0}/2! = \pi i$. The result remains the same when C is the circle $\{|z| = 2\}$, since these two contours contain the same singularities. □

(iii)

$$\int_C \frac{e^z}{(z^2 + z - 3/4)^2} dz = \int_C \frac{e^z}{(z+3/2)^2(z-1/2)^2} dz$$

$$= 2\pi i \frac{d}{dz}\left(\frac{e^z}{(z+3/2)^2}\right)|_{z=1/2} = 0$$

When the contour is changed, the result is also going to changed since the second contours includes one more singularity. Tedious computation shows the integral is $\pi i e^{-3/2}$. □

6. Proof: Just follow the hint. First of all, LHS of the equality in the hint is $2\pi i \times Res(f; i) = 2\pi i \times 1/(2i) = \pi$. The first term of RHS tends to $\int_{\infty}^{\infty} \frac{dx}{1+x^2}$ as R goes to ∞. To see the second term goes to 0 as R goes to ∞, we note

$$\left| \int_{c_R} \frac{dz}{1+z^2} \right| \leq \int_0^{\pi} \left| \frac{iRe^{i\theta}}{1+R^2 e^{2i\theta}} \right| d\theta \leq \int_0^{\pi} \frac{R}{R^2-1} d\theta \leq \pi \frac{R}{R^2-1} \longrightarrow 0$$

as R goes to ∞. □

1. Remark: The solutions here are omitted since it's sort of tedious computation. And the contours can always be chosen as $\{z : z \in \mathbb{R}, -R < z < R\} \cup \{z : z = Re^{i\theta}, 0 \leq \theta \leq \pi\}$. The answers for the integrals are $\pi/\sqrt{2}, 2\pi/3, \pi/3, \pi/2, \pi/2$. □

2a. Remark: The procedure is standard. So, I'll omit the complete proof, but point out the contour and integrand chosen in each problem and some tricky details. However, this doesn't mean they are the only feasible integrands and contours.
(1)The contour can be chosen as $\{z \in \mathbb{R} : -R < z < R, R > 0\} \cup \{z = Re^{i\theta} : 0 \leq \theta \leq \pi\}$. The integrand can be chosen as $\frac{e^{iz}}{z^2+1}$.
(2)The contour can be chosen as $\{z \in \mathbb{R} : -R < z < R, R > 0\} \cup \{z = Re^{i\theta} : 0 \leq \theta \leq \pi\}$. The integrand can be chosen as $\frac{ze^{iz}}{1+z^2}$. But you have to be very careful of the estimation: you might get something like

$$\int_0^{\pi} \frac{R^2 R^{-\sin\theta}}{R^2-1} d\theta$$

You cannot hastily claim this term goes to 0 as R goes to ∞, since the integrand is equal to $\frac{R^2}{R^2-1}$ at 0 and π, and the limit of this as R goes to ∞ is 1. However, we can still show the integral goes to 0 as R goes to ∞ by the following way: take any small positive number δ, and split up the integral into two parts

$$\int_0^{\pi} \frac{R^2 R^{-\sin\theta}}{R^2-1} d\theta = \int_{\delta}^{\pi-\delta} \frac{R^2 R^{-\sin\theta}}{R^2-1} d\theta + \int_{\theta\in[0,\delta]\cup[\pi-\delta,\pi]} \frac{R^2 R^{-\sin\theta}}{R^2-1} d\theta$$

The second term is smaller than 4δ, since the total length of the integration interval is 2δ while the integrand is no more than 2 for R large enough. As to the first term, recall the graph of $\sin\theta$ over the interval $[0, \pi]$ and you can see $\sin\theta \geq \sin\delta > 0$. Now, as R goes to $+\infty$, the integrand is tending to 0, since

$$\frac{R^2 R^{-\sin\theta}}{R^2-1} \leq \frac{R^2 R^{-\sin\delta}}{R^2-1}$$

And the latter one tends to 0 because of the influence of $R^{-\sin\delta}$.
(3)The contour can be chosen as $\{z \in \mathbb{R} : -R < z < R, R > 0\} \cup \{z = Re^{i\theta} : 0 \leq \theta \leq \pi\}$. The integrand can be chosen as $\frac{e^{iz}}{(z^2+1)^2}$.

3. Solution: As usual, we take the contour C as $\{z : z \in \mathbb{R}, -R < z < R\} \cup \{z : z = Re^{i\theta}, 0 \leq \theta \leq \pi\}$, and the function $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$. $f(z)$ has singularities $ai, -ai, bi$, and bi. Since the real parts of a and b are both positive, the singularities falling into C are ai and bi. So,

$$\int_C f(z)dz = 2\pi i\left[\frac{e^{iz}}{(z^2+a^2)(z+bi)}|_{z=bi} + \frac{e^{iz}}{(z^2+b^2)(z+ai)}|_{z=ai}\right] = \frac{\pi}{a^2-b^2}(\frac{e^{-b}}{b} - \frac{e^{-a}}{a})$$

We define $\{z : z \in \mathbb{R}, -R < z < R\}$ as II, and $\{z : z = Re^{i\theta}, 0 \leq \theta \leq \pi\}$ as I. And note

$$\left|\int_I f(z)dz\right| = \left|\int_0^\pi \frac{e^{i(R\cos\theta + i\sin\theta)}}{(R^2e^{2\theta i}+a^2)(R^2e^{2\theta i}+b^2)} iRe^{i\theta}d\theta\right|$$

$$\leq \int_0^\pi \frac{Re^{-R\sin\theta}}{(R^2-|a|^2)(R^2-|b|^2)}d\theta \leq \pi\frac{R}{(R^2-|a|^2)(R^2-|b|^2)}$$

It's clear that the last term goes to 0 as R goes to $+\infty$. So, we finally get

$$\frac{\pi}{a^2-b^2}(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}) = \int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = \int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$$

The last "=" is because $\frac{\sin x}{(x^2+a^2)(x^2+b^2)}$ is an odd function and it vanishes under the integration over the whole real line.□