MATH 418 Function Theory
Homework 1 Solution

Due January 30

Section 1.1

4. (5 points) Solution:
(i) $(1+i)^2 = 2i$;
(ii) $(1+i)^{11} = [(1+i)^2]^5(1+i) = -32+32i$;
(iii) $-1+2i$;
(iv) $(x^3 - 3xy^2) + (3x^2y - y^3)i$;
(v) $x^2 + y^2$;
(vi) $(x^2 - y^2 - 2xyi)/(x^2+y^2)$;
(vii) $[2x(1-y) + (x^2 - (y-1)^2)i]/[(1-y)^2 + x^2]$. □

5. (5 points) Proof: It suffices to note the following fact: $Re(1/z) = Re\frac{\bar{z}}{|z|^2} = \frac{1}{|z|^2}Rez$. □

10. (5 points) Proof: Multiply $\bar{\alpha}\bar{\beta}$ to both sides of $\alpha\beta = 0$, we then get $|\alpha|^2|\beta|^2 = 0$. So either $|\alpha| = 0$ or $|\beta| = 0$. □

13. (5 points) Proof: $P(z) - P(z_0) = \sum_{i=1}^{m} a_i(z^i - z_0^i)$. By problem 12, $P(z) - P(z_0)$ is divisible by $z - z_0$. □

Section 1.2

2. (5 points) Solution:
(a) The most convenient way of finding square root is using polar coordinate.
(i) $i = e^{\pi i/2}$. So $\sqrt{i} = e^{i(\pi/4+k\pi)}$, $k \in \mathbb{Z}$.
(ii) $\sqrt{-i} = e^{i(3\pi/4+k\pi)}$, $k \in \mathbb{Z}$.
(iii) $\sqrt{1+i} = (\sqrt{2}e^{i\pi/4})^{1/2} = 2^{1/4}e^{i(\pi/8+k\pi)}$, $k \in \mathbb{Z}$.
(iv) $\sqrt{2i+3} = (13)^{1/4}e^{i(\alpha/2+k\pi)}$, $k \in \mathbb{Z}$ and $\alpha = tan^{-1}\frac{2}{3}$.
(b) $z^2 + az + b = (z + a/2)^2 + b - a^2/4$. So we set $h = -a/2$. □

3. (5 points) Solution:
(a)
(i) y-axis;
(ii) Right half plane, not including the boundary;
(iii) Unit circle centered at origin;
(iv) Complement of closed unit disc;

(v) Straight line $y = 1$;
(vi) Half plane below straight line $y = 1$, not including boundary;
(vii) Open annulas enclosed by the unit circle and the circle of radius 2, center 0.

(b)
(i) Circle centered at 1 with radius 2;
(ii) Open disc with radius 0.01, centered at 1;
(iii) y-axis;
(iv) Boundary of a square with vertices 1, -1, i and -i.

□

5. (5 points) Proof: As in the hint, $962 = |\alpha|^2|\beta|^2 = |\alpha\beta|^2 = |-11+29i|^2 = 11^2 + 29^2$. □

10. (5 points) Proof:

(a) α is perpendicular to β if and only if $|\alpha-\beta|^2 = |\alpha|^2+|\beta|^2$ by Pythagoras' Theorem. This is equivalent to $Re(\alpha\bar{\beta}) = 0$ by calculation.

(b) $Re(i\alpha\bar{\alpha}) = 0$. So $i\alpha$ is perpendicular to α. And since $|i\alpha| = |\alpha|$, we conclude $i\alpha$ is a rotation of α. □

Section 1.3

4. (5 points) Proof: Suppose $z = |z|e^{i\theta}$, then $|z|^n(\cos n\theta + i\sin n\theta) = |z|^n e^{in\theta} = z^n = (|z|e^{i\theta})^n = |z|^n(\cos\theta + i\sin\theta)^n$. Let $|z| = 1$, we are done. □

7. (5 points)

(a) Proof: For sufficiency, note the given condition gives

$$\frac{|z_1 - z_2|}{|z_1' - z_2'|} = \frac{|z_1 - z_3|}{|z_1' - z_3'|} = \frac{|z_2 - z_3|}{|z_2' - z_3'|}$$

This implies the similarity between the two triangles. For necessity, note similarity implies corresponding angles of the two triangles are equal. So starting from

$$\frac{|z_1 - z_2|}{|z_1' - z_2'|} = \frac{|z_1 - z_3|}{|z_1' - z_3'|} = \frac{|z_2 - z_3|}{|z_2' - z_3'|}$$

we can actually conclude this sequence of equalities also holds when we remove the absolute value. □

(b) Proof:

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1' & z_2' & z_3' \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ z_1 - z_3 & z_2 - z_3 & z_3 \\ z_1' - z_3' & z_2' - z_3' & z_3' \end{vmatrix} = (z_1 - z_3)(z_2' - z_3') - (z_2 - z_3)(z_1' - z_3')$$

Similarly, we can get other equalities, which combine to give the condition in part (a).

□