MATH 418 Function Theory

Homework 10 Solution

Due April 17

Section 5.1

1. (3 points) Proof: The images of the original straight lines under ω become, respectively, t^2 and $e^{2\phi i}t^2$. Hence they are two straight lines through the origin meeting at an angle 2ϕ. So conformality needs not hold at points where $f'(z_0) = 0$. □

5. (10 points) Solution:
(a) $2z/(z+1)$.
(b) $\frac{z-1}{z+1}i$.
(c) z.
(d) iz.
(e) $\frac{z+i}{z-1}$. □

7. (3 points) Solution: ω maps $(-1, 0, 1)$ to three distinct points on the unit circle. So ω maps real axis to unit circle. Note $\omega(i) = 0$, so ω maps the upper half plane onto the unit circle. □

13. (3 points) Proof: Suppose $Tz = z$, then we get $cz^2 + dz = az + b$. This equation has two roots, if $c \neq 0$; only one root, if $c = 0$ and $d \neq a$; or infinitely many, if $c = 0$, $d = a$ and $b = 0$. The third case corresponds to identity mapping. So there are at most two fixed points unless T reduces to the identity transformation. □

14 (5 points) Proof: Let $\omega = f(z)$ be the solution of $X(\omega, \omega_1, \omega_2, \omega_3) = X(z, z_1, z_2, z_3)$. Then clearly $f(z)$ is a bilinear transformation by explicit computation. And it satisfies $f(z_i) = \omega_i$, $i = 1, 2, 3$. If there's another bilinear trasnformation $\omega = g(z)$ also satisfying $g(z_i) = \omega_i$, $i = 1, 2, 3$, then the bilinear transformation $f \circ g^{-1}(\omega)$ has three fix points. So $f \circ g^{-1} = id$. We conclude $f = g$. □

Section 5.2

10. (4 points) Proof: By solving equation $\omega = \frac{az+b}{cz+d}$, we find for any bilinear transformation g, it has an inverse and g^{-1} is also a bilinear transformation. By explicit computation, we can also see the composition of two bilinear trasnformations is still a bilinear transformation. So the class of all

bilinear transformations form a group. Furthermore, if g maps $\{|z| < 1\}$ onto itself, then g maps three distinct points on $|z| = 1$ to three distinct points on $|z| = 1$. If f is also such a mapping, $f \circ g$ maps thrr distinct points on $|z| = 1$ to three distinct points on $|z| = 1$. So $f \circ g$ is a bilinear transformation mapping $|z| < 1$ onto itself. This shows the set of all bilinear mappings of $|z| < 1$ onto itself forms a group. □

11. (5 points) Proof: The first part of the problem has been proved in the proof of problem 10. For the second part, just follow the hint, which is detailed enought. □

12. (5 points) Proof: For the first part, just imitate the proof for the second part of problem 11. For the second part, we have

$$Im\frac{az+b}{cz+d} = Im\frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2} = Im\frac{adz + bc\bar{z}}{|cz+d|^2} = \frac{ad-bc}{|cz+d|^2}y$$

Hence $Imz > 0$ iff $Im\frac{az+b}{cz+d} > 0$. It shows this class of mappings maps the upper half-plane onto itself and also the lower half-plane onto itself. □

Additonal Problems on page 409

9.1 (6 points) Solution:
(i) $(z^{1/2})^2$ determines an entire analytic function since it equals to $e^{\log z}$.
(ii) $(z^2)^{1/2}$ determines two entire functions, depending on the choice of square root.
(iii) $\cos z^{1/2} = \frac{1}{2}(e^{i\sqrt{z}} + e^{-i\sqrt{z}})$ determines an entire function since $\cos z = \cos(-z)$.
(iv) $(1-z)^{1/2}$ is a three-valued analytic function.
(v) $(e^z)^{1/3}$ determines three entire functions, depending on the choice of cubic root.
(vi) $(\cos z)^{1/2}$ determines a two-valued analytic function. □

9.3 (6 points) Proof: Since $zg'(z) = f(z)$, it suffices to show that $|z| = 1$ is a natural boundary for f. For this, just follow the hint, wihch is detailed enough. □