MATH 418 Function Theory
Homework 11 Solution

Due 04/24

Section 5.5

4. (9 points) Solution:
(i) $f(z) = 3z^3 + 2z^2 + z + c$, $f^*(z) = -3z^3 + 2z^2 - z + c$. So

$$h(z) = \frac{f - f^*}{f + f^*} = \frac{3z}{2} + \frac{2 - 3c}{4z^2 + 2c}z, h_1(z) = \frac{1}{2 - 3c}(4z + 2c/z)$$

So f is Hurwitz if and only if $4/(2 - 3c) \geq 0$ and $2c/(2 - 3c) > 0$, i.e. $0 < c < 2/3$. □
(ii) $f(z) = 4z^4 + z^3 + z^2 + c$, $f^*(z) = 4z^4 - z^3 + z^2 + c$. So

$$h(z) = \frac{z^3}{4z^4 + z^2 + c}, h_1(z) = 4z + 1/z + c/z^3$$

So f is Hurwitz only if $c = 0$. But then $f(0) = 0$. So f can't be Hurwitz. □
(iii) $f(z) = z^5 + 5z^4 + 4z^3 + 3z^2 + 2z + c$, $f^*(z) = -z^5 + 5z^4 - 4z^3 + 3z^2 - 2z + c$. So

$$h(z) = \frac{z}{5} + \frac{17z^3 + (10 - c)z}{5(5z^4 + 3z^2 + c)}, h_1(z) = \frac{5z}{17} + \frac{(1 + 5c)z^2 + 17c}{17z^3 + (10 - c)z}$$

$$h_2(z) = \frac{17z}{1 + 5c} + \frac{(10 - c)(1 + 5c) - 289c}{(1 + 5c)[(1 + 5c)z^2 + 17c]}z$$

$$h_3(z) = -\frac{(1 + 5c)^2}{5(c^2 + 48c - 2)} - \frac{17c(1 + 5c)}{5z(c^2 + 48c - 2)}$$

So f is Hurwitz if and only if $c^2 + 48c - 2 < 0$ and $c(1 + 5c) > 0$, i.e. $0 < c < \sqrt{578} - 24$. □

9. (3 points) Solution: In problem 4(i), let $c \geq 2/3$. □

10. (8 points) We skip over the solution since it's just tedious computation. □

14. (6 points) Proof: Let $z = iy$ with $y \in \mathbb{R}$, and assume

$$g(z) = \frac{\prod_{i=1}^{m}(z-\alpha_i)}{\prod_{j=1}^{n}(z-\beta_j)}\theta$$

then we have

$$\begin{aligned} g(z)g^*(z) &= \frac{\prod_{i=1}^{m}(z-\alpha_i)}{\prod_{j=1}^{n}(z-\beta_j)}\theta\frac{\prod_{i=1}^{m}(-z-\bar{\alpha}_i)}{\prod_{j=1}^{n}(-z-\bar{\beta}_j)}\bar{\theta} \\ &= \frac{\prod_{i=1}^{m}(z-\alpha_i)}{\prod_{j=1}^{n}(z-\beta_j)}\theta\frac{\prod_{i=1}^{m}(\bar{z}-\bar{\alpha}_i)}{\prod_{j=1}^{n}(\bar{z}-\bar{\beta}_j)}\bar{\theta} \\ &= g(z)\bar{g}(z) \\ &= 1 \end{aligned}$$

Since gg^* is rational, $g(z)g^*(z) = 1$ for any $z \in \mathbb{C}$. By unique factorization theorem for polynomials, $m = n$. And we can re-order α_i, β_j in such a way that $\alpha_i = -\bar{\beta}_i$. Hence g has the form in the problem. □

Section 5.6

3. (6 points)
(a) Proof: If $z_1/(1-z_1)^2 = z_2/(1-z_2)^2$, WLOG, assume z_1, $z_2 \neq 0$, then by solving this equation, we get

$$z_1 = 1/2[(z_2 + 1/z_2) + / - (z_2 - 1/z_2)] = z_2 or 1/z_2$$

Since $|z_1||z_2| < 1$, $z_1 = z_2$. □
(b) Proof: $z_1^2 = z_2^2$ if and only if $|z_1| = |z_2|$, and $e^{iargz_1} = e^{iargz_2}$. Choose z_1, z_2 in such a way that $|z_1| = |z_2|$, and $argz_1 = \pi/2$, $argz_2 = -\pi/2$. Then obviously $z_1 \neq z_2$. But since $2(argz_1 - argz_2) = 2\pi$, $z_1^2 = z_2^2$. □

7. (6 points) Solution:
$\omega = z + 1/z$: the Riemann surface is the same as that of example 6.3.
$\omega = z^2 - 1$: the Riemann surface is $\mathbb{C}$.
$z^2 = (\omega - 1)/\omega$: solve the equation, we get $\omega = 1/(1-z)(1+z)$. So the Riemann surface is the surface obtained by pasting two complex planes, eahc of which has a slit from -1 to 1. □

Section 5.7

6. (6 points) Proof: Let f be a simple function. By Theorem 7.4, if $f(\infty) = \infty$, then f is linear. If f maps $z_0 \neq \infty$ to ∞, consider $g = f(z_0 + 1/z)$. Then g is simple and $g(\infty) = \infty$. So $g(z) = a + bz$. This implies $f = [bz + (a - bz_0)]/(z - z_0)$. □

11. (6 points) Proof: $g(0) = 0$, $g'(0) = h'(0) - 1 = 0$. So 0 is a zero of g of degree at least 2. So $g(z)/z^2$ has a removable singularity and $z = 0$. Consider a circle centered at 0 with radius r. Then for $|z| \leq r$, by maximum principle, for r large enough

$$|\frac{g(z)}{z^2}| \leq \max_{|z|=r} |\frac{g(z)}{z^2}| \leq \frac{M+R}{r^2} = \frac{R^2}{Ar^2} \leq 1/A$$

So $|g(z)| \leq r^2/A$ on $|z| = r$. Meanwhile, on $|z| = r$, $|f(z)| \geq r - |\omega|$. So $|f(z)| > |g(z)|$ on $|z| = r$ if $r - |\omega| > r^2/A$. Set $r = A/2$, we get $|\omega| < A/4$ and $|z| = A/2$. By Rouché's theorem, $f(z)$ and $f(z) + g(z)$ have the same number of roots for $|z| < A/2$, i.e. one and only one solution of $h(z) = \omega$ exists for $|z| < A/2$. $\square$