MATH 418 Function Theory

Homework 2 Solution

Due February 6

Section 1.4

2. (2 points) Solution: $Rez > 1$, $0 < |z| < 1$, $Imz < 2|z|$, $|z-1| < |z+i|$ stand for domains. $|z| \leq 1$ is not a domain, since it's closed. And the last one, $2|z^2 - 1| < 1$, is not, sicne it's not connected (the existence of z^2 in the formula makes the pre-image consist of two sheaves). □

3. (3 points) Solution:
(i) $\omega = z^3$ maps Argz to 3Argz mod 2π. In order that the ω plane is just covered once, we should have $0 \leq \theta < 2\pi/3$.
(ii) $\omega = z^4$ maps Argz to 4Argz mod 2π. In order that the ω plane is just covered once, we should have $0 \leq \theta < \pi/2$.
(iii) $\omega = z^6$ maps Argz to 6Argz mod 2π. In order that the ω plane is just covered once, we should have $0 \leq \theta < \pi/3$. □

4. (6 points) Solution:
(i) $\omega = -z$: $g(\omega) = -\omega$, $R = \mathbb{C}$.
(ii) $\omega = 1/z$: $g(\omega) = 1/\omega$, $R = \mathbb{C} - \{0\}$.
(iii) $\omega = (1-z)/(1+z)$: $g(\omega) = (1-\omega)/(1+\omega)$, $R = \mathbb{C} - \{-1\}$.
(iv) $\omega = z^2$: $g(\omega) = \sqrt{|\omega|}e^{iArg\omega/2}$, $R = \{z : 0 \leq Argz < \pi\}$.
(v) $\omega = z^3$: $g(\omega) = |\omega|^{1/3}e^{iArg\omega/3}$, $R = \{z : 0 \leq Argz < 2\pi/3\}$.
(vi) $\omega = (z-1)^4 + i$: $g(\omega) = |\omega - i|^{1/4}e^{iArg(\omega-i)/4} + 1$, $R = \{z : 0 \leq Arg(z-1) < \pi/2\}$. □

Section 1.5

1. (5 points)

(a) Solution: nowhere; 0; -1; roots of $z^3 + 1 = 0$, i.e. -1, $(1-\sqrt{3}i)/2$ and $(1+\sqrt{3}i)/2$; roots of $z^4 - 16 = 0$, i.e. -2, 2, 2i and -2i; roots of $z^8 + z^5 - z^4 - z = 0$, i.e. 1, -1, i, -i, 0, $(1-\sqrt{3}i)/2$ and $(1+\sqrt{3}i)/2$. □

(b) Solution: $(z+1)/(z+1)$ has a removable discontinuity at -1 and it should be defined 1 there to remove the discontinuity. $(z^4 - z^2)/(z^8 + z^5 - z^4 - z)$ has removable discontinuity at 0 and 1. To remove the discontinuity, it should be defined 0 and 1/4, respectively. □

(c) Solution: Replace z with $1/\zeta$, we transform the above functions into the following ones: $1/\zeta$, ζ, $(1+\zeta^2)/(\zeta+\zeta^2)$, $2\zeta^3/(1+\zeta^3)$, $(1+\zeta)/(1+\zeta)$,

$(1+16\zeta^4)/(1-16\zeta^4)$, and $(\zeta^4-\zeta^6)/(1+\zeta^3-\zeta^4-\zeta^7)$. As $\zeta \to 0$, their values, respectively, has no limit in $\mathbb{C}$; has limit 0; has no limit in $\mathbb{C}$; has limit 0; has limit 1; has limit 1; has limit 0. □

5. (3 points) Solution: It seems reasonable to define $a^{\sqrt{2}}$ as the product $a \times a^{2/5} \times a^{1/100} \times a^{1/250} \times \dots$. But difficulties arise. First, we need to know if the product is convergent. Second, equation $z^n = z_0$ has n solutions and hence it's not clear what value should be assigned to $a^{1/5}$. Acutually, as we shall see later, $a^{\sqrt{2}}$ has infinitely many values. □

6. (2 points) The proof is exactly what the hint suggests, and is very easy. So we skip it over. □

Additional problems on chapter 1

1.2 (4 points) The proof is long, tedious, and easy. So I skip it over. □

1.4 (5 points) Proof: We work by induction.
For n=1, we choose $P_1(x) = x$. It's clearly unique.
Assume for $n \leq m$, the claim is true. Then for n=m+1, we have

$$\begin{aligned} z^{m+1} + \frac{1}{z^{m+1}} - (z+\frac{1}{z})^{m+1} &= -\sum_{k=1}^{m} C_{m+1}^k z^k \frac{1}{z^{m+1-k}} \\ &= -\sum [C_{m+1}^k z^k \frac{1}{z^{m+1-k}} + C_{m+1}^{m+1-k} z^{m+1-k} \frac{1}{z^k}] \\ &= \sum C_{m+1}^k [\frac{1}{z^{m+1-2k}} + z^{m+1-2k}] \end{aligned}$$

Since $|m+1-2k| \leq m$, by assumption, we have a unique polynomial Q_k such that $1/z^{m+1-2k} + z^{m+1-2k} = Q_k(z+1/z)$. Let $P_{m+1}(x) = x^{m+1} - \sum C_{m+1}^k Q_k(x)$. Then this is the desired polynomial for n=m+1. Uniqueness is shown in the proof as each Q_k is unique. □

2.1 (3 points) Proof:

$$\begin{aligned} (1-z)P(z) &= (a_0 + a_1 z + \dots + a_n z^n) - (a_0 z + a_1 z^2 + \dots + a_n z^{n+1}) \\ &= a_0 + (a_1 - a_0)z + \dots + (a_n - a_{n-1})z^n - a_n z^{n+1} \end{aligned}$$

So

$$|(1-z)P(z)| > a_0 - [(a_0 - a_1)|z| + \dots + (a_{n-1} - a_n)|z|^n - a_n z^{n+1}]$$

unless $z = 0$. If $|z| \leq 1$, then RHS$\geq a_0 - [(a_0 - a_1) + \dots + a_n] = 0$. So in this case, we have $|P(z)| > 0$, which means $P(z)$ can't have roots in the closed unit disc centered at origin. □

5.3 (7 points)

(a) Proof: Let $P(z) = \sum_{k=0}^{n} a_k z^k$ with $n \geq 1$, $a_n \neq 0$. Then $\lim_{|z|\to\infty} |P(z)| = \infty$. So outside some disc $\{z : |z| \leq R\}$, $|P(z)| > |a_0| = |P(0)|$. Therefore if the minimum of $|P(z)|$ for $|z| \leq R$ occurs at z_0, then $z = z_0$ gives the minimum of $|P(z)|$ with respect to the whole plane. □

(b) Proof: It's clear that $Q(z)$ is a polynomial. Since $Q(0) = 1$, we can see $Q(z)$ has the form of $1 + cz^m + \ldots$, with $m \geq 1$ and $\ldots$ are terms of higher degree. □

(c) Proof: If α is a root of $c\alpha^m = -1$, then $Q(\alpha z) = 1 + (\alpha z)^m c + \cdots = 1 - z^m + \ldots$. By definition of z_0, $|Q(z)| \geq 1$. So, $|Q(\alpha z)|$ obtains its minimum modulus with respect to the plane at $z = 0$. □

(d) Proof: let $z > 0$. Then as z small enough, $|Q(\alpha z)|$ is close to $1 - z^m/2$, which is smaller than 1 for z sufficiently small. Contradiction. □.