MATH 418 Function Theory
Homework 3 Solution

Due Febuarary 13, 2003

Section 2.1

2. (4 points) Solution: (a) Neither. (b) The second one. □

10. (4 points) Proof: Plug the expression of $\Delta\zeta$ into the first equation, we get

$$\Delta\omega = f'(\zeta)(g'(z)\Delta z + \varepsilon_2|\Delta z|) + \varepsilon_1|(g'(z)\Delta z + \varepsilon_2|\Delta z|)|$$

Since as $\Delta z \to 0$, $\Delta\zeta \to 0$, we conclude $\Delta z \to 0$ implies $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. Hence

$$\Delta\omega = f'(g(z))g'(z)\Delta z + o(|\Delta z|)$$

Let $\Delta z \to 0$, we are done. □

Section 2.2

7. (6 points) Proof:

$$\begin{aligned} & \cos(z_1 + z_2) + i\sin(z_1 + z_2) \\ = \; & e^{z_1+z_2} \\ = \; & e^{z_1}e^{z_2} \\ = \; & (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) \\ = \; & (\cos z_1\cos z_2 - \sin z_1\sin z_2) + i(\cos z_1\sin z_2 + \sin z_1\cos z_2) \qquad (1) \end{aligned}$$

Replace z_1, z_2 with $-z_1$, $-z_2$ in the above equation, we get

$$\begin{aligned} & \cos(z_1 + z_2) - i\sin(z_1 + z_2) \\ = \; & (\cos z_1\cos z_2 - \sin z_1\sin z_2) - i(\cos z_1\sin z_2 + \sin z_1\cos z_2) \qquad (2) \end{aligned}$$

Divide (1)-(2) by 2i, we get

$$\sin(z_1 + z_2) = \frac{2i(\sin z_1\cos z_2 + \cos z_1\sin z_2)}{2i} = \sin z_1\cos z_2 + \cos z_1\sin z_2$$

Divide (1)+(2) by 2, we get

$$\cos(z_1 + z_2) = \frac{2(\cos z_1\cos z_2 - \sin z_1\sin z_2)}{2} = \cos z_1\cos z_2 - \sin z_1\sin z_2$$

□

13. (3 points) Proof:

$$\begin{aligned} & 2\sqrt{2}e^{\pi i/12} \\ = \; & 2\sqrt{2}e^{\pi i/3}e^{-\pi i/4} \\ = \; & 2\sqrt{2}(1/2+\sqrt{3}/2i)(\sqrt{2}/2-\sqrt{2}/2i) \\ = \; & (1+\sqrt{3}i)(1-i) \\ = \; & (1+\sqrt{3})+(\sqrt{3}-1)i \end{aligned}$$

□

15. (4 points) Proof: $\omega = e^{(1+i)t} = e^t e^{it}$. So, when t goes from $-\infty$ to ∞, $\mathrm{Im}\omega = e^{it}$ repeats the value on the unit circle centered at origin, and $\mathrm{Re}\omega = e^t$ gets larger and larger. So the graph of ω is a logarithmic spiral. □

Section 2.3

2. (3 points) Proof: Since $e^{\mathrm{Log}1+i\pi/2} = e^{\pi i/2} = i$ and $0 \le \pi/2 < 2\pi$, we get by definition $\mathrm{L}ogi = \pi i/2$. Hence

$$i^i = e^{i\log i} = e^{i(\mathrm{L}ogi+2\pi ik)} = e^{i(i\pi/2+i2\pi k)} = e^{-(4k+1)\pi/2}$$

where $k \in \mathbb{Z}$. □

7. (12 points) Solution:

(i) $\omega = \cos^{-1} z$ implies $\cos\omega = z$, i.e. $(e^{i\omega}+e^{-i\omega})/2 = z$. Hence $e^{2i\omega} - 2ze^{i\omega}+1 = 0$. Solve this equation, we get

$$e^{i\omega} = \frac{2z+\sqrt{4z^2-4}}{2} = z+\sqrt{z^2-1}$$

Note here square roots are allowed to take two values, so it suffices to take only one of the roots of the quadratic equation. Since $[z-(z^2-1)^{1/2}][z+(z^2-1)^{1/2}] = 1$, we conclude $z+(z^2-1)^{1/2} \neq 0$. So $\log[z+(z^2-1)^{1/2}]$ is well defined and $w = -i\log[z+(z^2-1)^{1/2}]$. □

(ii) $\omega = \tan^{-1} z$ implies

$$z = \tan\omega = -i\frac{e^{i\omega}-e^{-i\omega}}{e^{i\omega}+e^{-i\omega}}$$

So we get

$$e^{2i\omega} = \frac{i-z}{i+z}$$

Take logarithm, we get

$$\omega = \frac{i}{2}\log\frac{i+z}{i-z}$$

$\square$

(iii) $\omega = \cosh^{-1} z$ implies

$$z = \frac{e^{i(i\omega)} + e^{-i(i\omega)}}{2} = \frac{e^{\omega} + e^{-\omega}}{2}$$

So $e^{2\omega} - 2ze^{\omega} + 1 = 0$. Solve this equation and take only one of the roots since square roots are allowed to take two values

$$e^{\omega} = z + \sqrt{z^2 - 1}$$

So we get $\omega = \log[z + (z^2 - 1)^{1/2}]$. $\square$

(iv) $\omega = \tanh^{-1} z$ implies $z = \frac{e^{\omega} - e^{-\omega}}{e^{\omega} + e^{-\omega}}$. So $z(e^{2\omega} + 1) = e^{2\omega} - 1$. Simplify this equation, we get $e^{2\omega} = \frac{1+z}{1-z}$. Hence $\omega = \frac{1}{2}\log\frac{1+z}{1-z}$. $\square$

8. (4 points) Proof: Since $tan^{-1}z = \frac{i}{2}\log\frac{i+z}{i-z}$,

$$\frac{d}{dz}\tan^{-1} z = \frac{i}{2}\frac{i-z}{i+z}\frac{i-z+(i+z)}{(i-z)^2} = \frac{1}{z^2+1}$$

Since $\sinh^{-1} z = \log[z + (z^2+1)^{1/2}]$, we get

$$\frac{d}{dz}\sin h^{-1} Z = \frac{1 + \frac{1}{2}(z^2+1)^{-1/2}2z}{z + (z^2+1)^{1/2}} = \frac{1 + \frac{z}{\sqrt{z^2+1}}}{z + \sqrt{z^2+1}} = \frac{1}{\sqrt{z^2+1}}$$

$\square$

Chapter 2 Additional Problems

1.3 (4 points) Proof: We note the following relations

$$\sum_{i=1}^{n} \alpha_i = \frac{\text{the coefficient of } z^{n-1} \text{ in P(z)}}{-1 \times \text{the coefficient of } z^{n} \text{ in P(z)}}$$

and

$$\sum_{j=1}^{n-1} \beta_i = \frac{\text{the coefficient of } z^{n-2} \text{ in } P'(z)}{-1 \times \text{the coefficient of } z^{n-1} \text{ in } P'(z)}$$

Since the coefficient of z^{n-2} in $P'(z) = (n-1)\times$ the coefficient of z^{n-1} in P(z), and the coefficient of z^{n-1} in $P'(z) = n\times$the coefficient of z^n in P(z), we can conclude the desired equation. $\square$

1.4 (6 points)

(a) Solution: $\text{Re}(\alpha z + \beta)$ has the form of $ax + by + c$ where x and y are the real and imaginary parts of z, respectively. So $\text{Re}(\alpha z + \beta) < 0$ if and only if $ax + by + c < 0$ and therefore stands for a half plane. $\square$

(b) Proof: z is a root of Q(z) if and only if $\tilde{z}$ is a root of $P(\tilde{z})$. So the zeros of Q lie in the half plane $\text{Re}(\alpha z + \beta) < 0$ if and only if the zeros of P have negative real parts. By Example 1.3, all the zeros of $P'(\tilde{z})$ also have negative real parts. But $P'(\tilde{z}) = Q'(z)\frac{dz}{d\tilde{z}}$. So $\tilde{z}$ is a root of $P'(\tilde{z})$ if and only if z is a root of $Q'(z)$. So by the correspondence between z and $\tilde{z}$, all the zeros of $Q'(z)$ lie in the half place $\text{Re}\tilde{z} < 0$. $\square$

(c) Proof: Clear from (b). $\square$