MATH 418 Function Theory
Homework 1 Solution

Due February 20

Section 2.4

1. (8 points) Solution:
(1) $\omega = z$: The fundamental region is $\mathbb{C}$.
(2) $\omega = z^2$: The fundamental region is $\{z : 0 \leq Argz < \pi\}$.
(3) $\omega = z^{1/4}$: Riemann surface has four sheets and has a structure similar to that of $z^{1/3}$ in Example 4.1.
(4) $\omega = z^{3/2}$: Fundamental region is $\{z : 0 \leq Argz < 2\pi/3\}$. Riemann surface for z^3 has a structure similar to that of $z^{1/3}$ in Example 4.1, but only two sheets.
(5) $\omega = z^e$: $\omega = e^{e \log z}$, so Riemann surface is similar to that of $\log z$ but has infinitely many sheets, never closing to itself. Fundamental region is $\{z : 0 \leq Argz < 2\pi/e\}$.
(6) $\omega = \log(1-z)$: Riemann surface is similar to that of $\log z$, but centered at $(1, 0)$.
(7) $\omega^2 = \log z$: It's the composition of the Riemann surface of a logarithm and that of a square root function. □

3. (4 points) Solution:
(1) $z^{\alpha+\beta}$ requires $\log 1 = 0$.
(2) $\log z^\alpha = \alpha \log z$ requires $\log 1 = 0$.
(3) By Example 3.1, this is always true and we don't need $\log 1 = 0$.
(4) $\log(z^\alpha)^\beta = \beta \log(z^\alpha) = \beta\alpha \log z$ requires $\log 1 = 0$.

□

4. (3 points) Proof: $\phi'(z) = a/(az) - 1/z = 0$. So ϕ is constant. Let $\phi = z_0$, then $z_0 = \phi(1) = \log a$. This shows $\log(az) = \log a + \log z$. □

Section 2.5

1. (4 points) Solution:
(1) $\Delta(x^2 - y^2 + 1) = 1 - 1 = 0$. Harmonic.
(2) $\Delta = (x^3 - y^3) = 6x - 6y \neq 0$. Not harmonice.
(3) $\Delta(3x^y - y^3 + xy) = 6y - 6y = 0$. Harmonic.
(4) $\Delta(x^5 - 6x^2y^2 + y^4 + x^3y - xy^3) = 12x^2 - 12y^2 + 6xy - 12x^2 + 12y^2 - 6xy = 0$. Harmonic. □

8. (5 points) Proof: Suppose $f = f_1 + if_2$ and $g = g_1 + ig_2$, then

$$\frac{\partial(f_1 + g_1)}{\partial x} = \frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial x} = \frac{\partial f_2}{\partial y} + \frac{\partial g_2}{\partial y} = \frac{\partial(f_2 + g_2)}{\partial y}$$

Similarly we can show $\frac{\partial(f_1+g_1)}{\partial y} = -\frac{\partial(f_2+g_2)}{\partial x}$. So $f + g$ satisfies the Cauchy-Riemann equations.

By similar argument, we can show fg also satisfies the Cauchy-Riemann equations.

$a + bi$ and $x + yi$ obviously satisfy Cauchy-Riemann equations at every point since a, b are constant functions and $\partial x/\partial x = 1 = \partial y/\partial y$, $\partial x/\partial y = 0 = -\partial y/\partial x$.

Hence we can deduce the polynomials P(z) satisfies the Cauchy-Riemann equations. □

9. (4 points) Proof: By definition, $\text{Im} z^{-2} = -2xy/(x^2 + y^2)$. Therefore

$$\phi(x, y) = \begin{cases} \frac{-2xy}{x^2+y^2}, & for (x, y) \neq (0, 0) \\ 0, & for (x, y) = (0, 0) \end{cases}$$

When $(x, y) \to (0, 0)$, $\phi(x, y)$ doesn't have limit, as can be seen by letting $y = kx$, for varying k.

To check ϕ satisfies Laplace's equation, note ϕ is symmetric in x and y, except a sign. So $\partial^2\phi/\partial x^2 = -\partial^2/\partial y^2 \phi$, i.e. $\Delta\phi = 0$. □

Section 2.6

1. (7 points) Solution:
(1) $V = 1$, $|V| = 1$, $\phi = x$, $\psi = y$.
(2) $V = -i$, $|V| = 1$, $\phi = -y$, $\psi = x$.
(3) $V = 1 - i$, $|V| = \sqrt{2}$, $\phi = x - y$, $\psi = x + y$.
(4) $V = -1/\bar{z}^2$, $|V| = 1/|z|^2$, $\phi = x/(x^2 + y^2)$, $\psi = -y/(x^2 + y^2)$.
(5) $V = i/\bar{z}^2$, $|V| = 1/|z|^2$, $\phi = y/(x^2 + y^2)$, $\psi = x/(x^2 + y^2)$.
(6) $V = 2\bar{z}$, $|V| = 2|z|$, $\phi = x^2 - y^2$, $\psi = 2xy$.
(7) $V = -2i\bar{z}$, $|V| = 2|z|$, $\phi = -2xy$, $\psi = x^2 - y^2$. □

4. (4 points) Proof:

$$V = \bar{f}'(z) = \frac{\bar{\alpha}}{\bar{z}} = \frac{\bar{\alpha} z}{|z|^2} = \frac{\bar{\alpha} r e^{i\theta}}{r^2} = \frac{\bar{\alpha} e^{i\theta}}{r}$$

So $arg \frac{V}{z} = arg \frac{\bar{\alpha}}{r^2}$ (constant). Also, if we let $\alpha = a + bi$, then $f(z) = (a + bi)(\log r + i\theta) = a \log r - b\theta + (b \log r + a\theta)i$. Then streamlines are defined by $b \log r + a\theta = c$, i.e. $r = e^{(c-a\theta)/b}$. So they're logarithmical spirals. □

6. (6 points) Proof: $f(z) = \log(z-1) - \log(z+1)$. Then the potential corresponding to $f(z)$ can be seen as a superposition of a source at 1 and a sink at -1.

$$V = \bar{f}'(z) = \frac{1}{z-1} - \frac{1}{z+1} = \frac{2}{z^2-1}$$

So $|V| = 2/|\bar{z}^2 - 1|$, i.e. $|V||\bar{z}^2 - 1| = 2$.

The streamlines have equations of the form $Arg\frac{z-1}{z+1} = C$. This shows vector AP and PB have constant angles in between, where $A = (-1, 0)$, $P = z$ and $B = (1, 0)$. These are circles. □

Chapter 2 Additional Problems

1.5 (5 points) Proof:

$$\frac{f(z) - f(0)}{z - 0} = \frac{x^2 y}{x^6 + y^2}$$

If $y = kx$ with $k \neq 0$, then

$$\frac{x^2 y}{x^6 + y^2} = \frac{kx^3}{x^6 + k^2 x^2} = \frac{kx}{x^4 + k^2} \to 0$$

as $x \to 0$. If $x = ky$, then

$$\frac{x^2 y}{x^6 + y^2} = \frac{k^2 y^3}{k^6 y^6 + y^2} = \frac{k^2 y}{k^6 y^4 + 1} \to 0$$

as $y \to 0$. Hence the limit defining $f'(0)$ exists, as $z \to 0$ along any straight line passing through the origin. However, f is not continuous at $z = 0$. To see this, let $y = kx^3$, then $f(z) \to k/(1 + k^2)$ as $z \to 0$. This is dependent on the choice of k. So the limit is indefinite. □