MATH 418 Function Theory
Homework 5 Solution

Due Febuarary 27, 2003

Section 3.1

1. (5 points) Proof: Note the origin is not on C, so applying Example 1.2 directly, we conclude $\int_C z^n dz = 0$, for $n \le -2$. Note this is also straightforward by Cauchy's formula (for derivatives of an anlytic function.)

If $n = -1$, and C is the circle $\{z : |z| = 1\}$, we have

$$\int_C z^{-1} dz = \int_0^{2\pi} e^{-it} i e^{it} dt = 2\pi i$$

$\square$

3. (3 points) Proof: Apply Theorem 1.1, we get $f(z_1) = f(z_2), \forall z_1, z_2 \in D$. So f is a constant. $\square$

9. (5 points) Proof: By the chain rule for real functions, we get

$$\phi'(t) = u_x(\xi,\eta)\xi'(t) + u_y(\xi,\eta)\eta'(t) + i[v_x(\xi,\eta)\xi'(t) + v_y(\xi,\eta)\eta'(t)]$$

Then use Cauchy-Riemann equations, we get

$$\begin{aligned}
\phi'(t) &= u_x(\xi,\eta)\xi'(t) - v_x(\xi,\eta)\eta'(t) + i[v_x(\xi,\eta)\xi'(t) + u_x(\xi,\eta)\eta'(t)] \\
&= [u_x(\xi,\eta) + iv_x(\xi,\eta)]\xi'(t) + i[iv_x(\xi,\eta) + u_x(\xi,\eta)]\eta'(t) \\
&= F'(\zeta(t))(\xi'(t) + i\eta'(t)) \\
&= F'(\zeta(t))\zeta'(t)
\end{aligned}$$

$\square$

10. (3 points) Proof: For $\zeta(t)$,

$$x(t) = \begin{cases} t^2, & t \in [-1,0] \\ 0, & t \in [0,1] \end{cases}$$

$$y(t) = \begin{cases} 0, & t \in [-1,0] \\ t^2, & t \in [0,1] \end{cases}$$

Then obviously x, $y \in C^1[-1,1]$. So $\zeta(t)$ is an arc. $\square$

12. (6 points) Proof: $\xi(t) = -t$ for $t \in [-1, 0]$ is an arc since Reξ, Im$\xi \in C^1[-1, 0]$, by $\xi'(t) = -1$. And $\eta(t) = t + it^3 \sin(1/t)$ is also an arc since $\eta'(t) = 1 + i(3t^2 \sin(1/t) - t\cos(1/t)) \in C[0, 1]$. They're the desired simple arcs (it's easy to see $\eta(t)$ has no self-intersection and drawing a picture is helpful to seeing why these two arcs have infinitely many isolated intersections. We shall skip over the analytic proof, since it's trivial and tedious.) Combine these two arcs, we get the desired contour. □

Section 3.2

1. (6 points) Solution: The unit circle has the parametric representation $z = e^{i\theta}, 0 \le \theta \le 2\pi$. So,
$\int_C \frac{dz}{z} = \int_0^{2\pi} i d\theta = 2\pi i$.
$\int_C \frac{dz}{|z|} = \int_C dz = 0$.
$\int_C \frac{|dz|}{z} = \int_0^{2\pi} e^{-i\theta} d\theta = 0$.
$\int_C \frac{dz}{z^2} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{2i\theta}} d\$ \int_0^{2\pi} ie^{-i\theta} d\theta = -e^{-i\theta}|_0^{2\pi} = 0$.
$\int_C \frac{dz}{|z^2|} = \int_C dz = 0$.
$\int_C \frac{|dz|}{z^2} = \int_0^{2\pi} e^{-2i\theta} d\theta = 0$.

□

7. (5 points) Solution: Suppose C is parametrized by $(x(t), y(t))$, $t \in [a, b]$. Then

$$\begin{aligned}\int_C f(z)dz &= \int_a^b [u(x,y)x'(t) - v(x,y)y'(t)]dt + i\int_a^b [u(x,y)y'(t) + v(x,y)x'(t)]dt \\ &= \int_a^b [u(x,y)\frac{dz}{dt} + iv(x,y)\frac{dz}{dt}]dt \\ &= \int_a^b f(z(t))z'(t)dt\end{aligned}$$

This is the definition given in section 1. □

8. (5 points) Proof: Let $F(t) = \int_a^t f(\zeta_1(s))\zeta_1'(s)ds$, $F(\tau) = \int_\alpha^\tau f(\zeta_2(\sigma))\zeta_2'(\sigma)d\sigma$. Then

$$\frac{dF_1}{dt} = f(\zeta_1(t))\zeta_1'(t), \frac{dF_2}{d\tau} = f(\zeta_2(\tau))\zeta_2'(\tau)$$

So $\zeta_1(t) = \zeta_2(\tau)$ implies $\zeta_1'(t) = \zeta_2'(\tau)\phi'(t)$. So

$$\frac{dF_2}{dt} = \frac{dF_2}{d\tau}\phi'(t) = \frac{dF_1}{dt}$$

Since $F_1(a) = 0 = F_2(\alpha)$, $F_1 = F_2$.

□

Section 3.3

2. (6 points) Proof: Let $C = \{z : |z| = r\}$ with $r < R$ and $k(\theta) = R^2 - 2Rr\cos\theta + r^2$. Then

$$\int_C \frac{dz}{R-z} = 0$$

Meanwhile, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\frac{R\cos\theta d\theta}{k(\theta)} &= \frac{1}{2\pi}\int_0^{2\pi}(-\frac{1}{2r}+\frac{R^2+r^2}{2rk(\theta)})d\theta \\
&= -\frac{1}{2r}+\frac{1}{2\pi}\frac{R^2+r^2}{2r(R^2-r^2)}\int_0^{2\pi}\frac{R^2-r^2}{k(\theta)}d\theta \\
&= \frac{r}{R^2-r^2}
\end{aligned}$$

The last equality is due to Example 3.1. □

3. (6 points) Proof: WLOG, we assume $a > 0$. Take the contour C as the rectangle with vertices at $-b$, b, $b+ia$ and $-b+ia$, where b is a positive number. We take the orientation of C as clockwise. Let $f(z) = e^{-z^2}$, then $\int_C f(z)dz = \int_{-b}^{b} f(x+ia)dx + \int_a^0 f(b+iy)dy + \int_b^{-b} f(x)dx + \int_0^a f(-b+iy)dy$. We claim $\int_a^0 f(b+iy)dy$ and $\int_0^a f(-b+iy)dy$ go to 0, as $b \longrightarrow +\infty$. Indeed, we have

$$|\int_a^0 f(b+iy)dy| \leq \int_0^a |f(b+iy)|dy = \int_0^a |e^{-b^2}e^{-2byi}e^{y^2}|dy$$

$$= \int_0^a e^{-b^2}e^{y^2}dy \leq e^{-b^2}ae^{a^2} \overset{b\to+\infty}{\longrightarrow} 0$$

Similarly, $\int_0^a f(-b+iy)dy \longrightarrow 0$ as $b \longrightarrow +\infty$. Since $f(z)$ is analytic in the whole plane, $\int_C f(z)dz = 0$. Hence if we let $b \longrightarrow +\infty$, we have $0 = I(a) + \int_{\infty}^{-\infty} f(x)dx$. So, $I(a) = \int_{-\infty}^{\infty} f(x)dx$(constant independent of a). □