MATH 418 Function Theory
Homework 6 Solution

Due March 6

Section 3.4

3. (4 points) Proof: Let $P(z) = a(z - z_1) \dots (z - z_n)$. Then

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_n}$$

Since

$$\frac{1}{2\pi i} \int_C \frac{dz}{z - a} = \begin{cases} 0, & \text{if a is outside C} \\ 1, & \text{if a is inside C} \end{cases}$$

We conclude

$$\frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z)} dz = \text{number of roots of P(z) inside C}$$

□

Section 3.5

1. (4 points) Proof: Let $g(z) = e^{f(z)}$, then $|g(z)| \leq e^M$. By Liouville's Theorem, g is constant. Hence so is f. □

2. (5 points) Proof: By formula (5.6), $f^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{\zeta^{n+1}}$, where $C = \{z : |z| = R\}$. So

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(Re^{i\theta}) Rie^{i\theta} d\theta}{R^{n+1} e^{i(n+1)\theta}}$$

and hence

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi R^n} \int_0^{2\pi} |f(Re^{i\theta})| d\theta$$

□

3. (6 points) Proof: $\forall \alpha \in \mathbb{C}$, choose $D_R = \{z : |z - \alpha| < R\}$. Then by Cauchy's formula (5.6)

$$
\begin{aligned}
|f^{(m+1)}(\alpha)| &\leq \frac{(m+1)!}{2\pi}\int_0^{2\pi}\left|\frac{f(\alpha+Re^{i\theta})Rie^{i\theta}}{R^{m+2}e^{i(m+2)\theta}}\right|d\theta \\
&= \frac{(m+1)!}{2\pi}\int_0^{2\pi}\frac{|f(\alpha+Re^{i\theta})|}{R^{m+1}}d\theta \\
&\leq \frac{(m+1)!}{2\pi R^{m+1}}\int_0^{2\pi}|\alpha+Re^{i\theta}|^m M d\theta \\
&\leq \frac{(m+1)!M(|\alpha|+R)^m}{R^{m+1}}
\end{aligned}
$$

where we take R sufficiently large such that $|\alpha + Re^{i\theta}| \geq R - |\alpha|$ is very large and we can use the condition $|f(z)| \leq M|z|^m$ for $z = Re^{i\theta} + \alpha$ $(0 \leq \theta \leq 2\pi)$. Let $R \to \infty$, in the above inequality and we get $f^{(m+1)}(\alpha) = 0$. □

9. (6 points) Proof: We prove the general case. $\forall z_0 \in \mathbb{C}$,

$$
\begin{aligned}
\left|\frac{F(z)-F(z_0)}{z-z_0} - i\int_C e^{itz_0}tf(t)dt\right| &\leq \int_C\left|\frac{e^{izt}-e^{iz_0t}}{z-z_0} - ie^{itz_0}t\right||f(t)|dt \\
&\leq L\int_C |z-z_0||f(t)|dt
\end{aligned}
$$

for some constant L, for z sufficiently close to z_0. Let $z \to z_0$, we are done. □

Section 3.6

1. (4 points) Proof: To show the first case, apply Theorem 6.1 directly. To show the second case, imitate the proof of Example 6.1. □

3. (5 points) Proof: Just follow the hint and work by induction. □

4. (6 points) Proof: First, the functions thus defined are continuous in a small neighbourhood D of 0, and analytic in $D - \{0\}$. Let C be any triangle such that $\bar{C} \subset D$. Then $\int_C f(z)dz = 0$, where f stands for any of the three functions . This is because if 0 is inside C, we then divide C into three smaller triangles, with 0 as a vertex. f is analytic in the three smaller triangles and hence produces 0 by integration and Cauchy's theorem. Then use Morera's Theorem, and we are able to conclude f is analytic near 0.

By use power series expansion of the functions, we get

$$S(z) = z - \frac{z^3}{3\times 3!} + \frac{z^5}{5\times 5!} - \dots,$$

$$E(z) = z + \frac{z^2}{2\times 2!} + \frac{z^3}{3\times 3!} + \dots$$

$$L(z) = z - \frac{z^2}{2 \times 2} + \frac{z^3}{3 \times 3} - \dots$$

$\square$

11. (6 points) I guess the hint is detailed engouth. So I would like to skip over the solution. (Man! The hints in this book are so detailed, it's kind of cheating!!.) $\square$

12 (4 points) Proof: f and g are analytic in $\{|z| < R\}$. So fg is analytic in $\{|z| < R\}$. Let c_n be the coefficient of z^n in the Taylor series of fg. Then

$$\begin{aligned}
c_n &= \frac{1}{n!}\frac{d^n}{dz^n}[f(z)g(z)] \\
&= \frac{1}{n!}\sum_{k=0}^{n} C_n^k \frac{d^k}{dz^k} f(z) \frac{d^{n-k}}{z^{n-k}} g(z) \\
&= \frac{1}{n!}\sum a_k b_{n-k} k!(n-k)! C_n^k \\
&= \sum_{k=1}^{n} a_k b_{n-k}
\end{aligned}$$

And the Taylor series of fg converges uniformly to $f(z)g(z)$ in $\{|z| \leq \rho < R\}$. $\square$