MATH 418 Function Theory
Homework 7 Solution

Due March 27

Section 3.7

2. (4 points) Proof: Since f is analytic in G and $f \neq 0$ in G, $1/f$ is analytic in G. By Theorem 7.5, $1/|f|$ cannot have a maximum value anywhere in G unless f is a constant. So $1/|f|$ assumes its maximum value on ∂G, i.e. $|f|$ assumes its minimum value on ∂G. $\square$

5. (6 points) Proof: Let $g = e^h$. Then g is analytic in D and is not constant. So $|g|$ doesn't attain maximum in D. By Problem 2, $|g|$ doesn't attain minimum in D either. Since $|g| = e^{Reh}$, this means Reh attains neither a maximum nor a minimum in D.

Let f and g be analytic in the bounded domain D. Set $h = f - g$. Then $Reh = 0$ on ∂D. By part one, this implies $h =$constant. So $f - g = ic$, for some constant c. $\square$

7. (6 points) Proof: Since f is analytic for $|z| \leq R$, and $f(0) = 0$, we can write f as $f(z) = zg(z)$, where $g(z)$ is analytic for $|z| \leq R$. On the boundary $\{|z| = R\}$, $|g(z)| = |f(z)/z| \leq M/R$. By maximum principle, $|g(z)| \leq M/R$ in $\{|z| < R\}$. So $|f(z)| \leq |g(z)||z| \leq |z|M/R$. $\leq$ becomes $<$ in $\{|z| < R\}$, unless $g =$ constant. $\square$

Section 3.8

1. (7 points) Solution: This kind of problem can be solved quite easily by looking at a function's Laurent series. Unfortunately, the most useful theorem is not in section 8, but in section 9 (Theorem 9.4, page 167). Of course, straightforward observation is also beneficial in some cases.

(1) e^z: e^z is analytic everywhere in $\mathbb{C}$. To judge the property of ∞, we consider all the three possibilities. First, ∞ cannot be removable by problem 3, since e^z is not a constant function. Second, ∞ is not a pole, since $|e^{in}| \leq 1$, no matter how large $n \in \mathbb{N}$ is. So ∞ is an essential singularity.

(2) $\frac{\cos z}{z}$: Only 0 or ∞ could have problems since $\frac{\cos z}{z}$ is analytic elwhere. Note the Laurent series of $\frac{\cos z}{z} = \sum_{n=2k}^{\infty} \frac{(iz)^n}{zn!}$, where $k \in \mathbb{N} \cup \{0\}$, we conclude 0 is a pole by Theorem 9.4, since $1/z$ appears in the series. To to see the property of ∞, replace z with $1/\zeta$, it's clear that $\zeta = 0$ is an essential singularity, by Theorem 9.4. Hence ∞ is an essential singularity of $\frac{\cos z}{z}$.

(3) $\frac{e^z-1}{z(z-1)}$: 1 is a pole and 0 is removable. Replace z with $1/\zeta$, we get $(e^{\frac{1}{\zeta}} - 1)\frac{\zeta^2}{1-\zeta}$. It's clear that this function is not differenctiable at $\zeta = 0$(argue by direct computation according to the definition of being analytic). So, ∞ cannot be removable. Furthermore, if $z \longrightarrow \infty$ along the negative x-axis, then $\frac{e^z-1}{z(z-1)}$ goes to 0. So, ∞ cannot be a pole. Hence, ∞ has to be an essential singularity.
(4) $\frac{z^2-1}{z^2+1}$: This function is equal to $1 - \frac{2}{z^2+1}$. So, i and $-i$ are two poles. Replace z with $1/\zeta$, we get $\frac{1-\zeta^2}{1+\zeta^2}$. This new function is differentiable at $\zeta = 0$. So, ∞ is a removable singularity of the original function.
(5) $\frac{z^5}{z^3+z}$: By similar arguement, i and $-i$ are two poles. 0 is a removable singularity. And ∞ is also a pole, since after replacing z with $1/\zeta$, we get $\frac{1}{\zeta^2+\zeta^4}$.
(6) $e^{\cosh z}$: $\cosh z$ is an entire function, so is e^z. Since the function under consideration is the composition of two entire functions, it's entire. To judge ∞, note first by problem 3, ∞ cannot be removable. Let $z = in$ where n is just a natural number. Then we can see, as $n \longrightarrow +\infty$, hence $z \longrightarrow \infty$, $e^{\cosh z}$ is bounded. So, ∞ cannot be a pole. So, ∞ has to be an essential singularity.
(7) $\frac{z(z-\pi)^2}{(\sin z)^2}$: We first solve the equation $e^{iz} = e^{-iz}$ and get solutions $z = k\pi$ where $k \in \mathbb{Z}$. For $k \neq 1, 0$, $k\pi$ becomes a pole since $\sin z = 0$ here. For 0, as $z \longrightarrow 0$, $\frac{\sin z}{z} \longrightarrow 1$, by the definition of the derivative of $\sin z$ at 0. So, 0 is a pole. For π, note $\sin(z - \pi) = -\sin z$, we again return to the previous case. But this time the dominator $\sin(z - \pi)$ and the nominator $(z - \pi)$ have the same power. So, $\frac{z(z-\pi)^2}{(\sin z)^2} \longrightarrow \pi$ as $z \longrightarrow \pi$. Hence, π is a removable singularity. Let $z \longrightarrow \infty$ along the positive x-axis, the function has no limit. So, ∞ cannot be a pole or removable. So, it's an essential singularity. $\square$

3. (5 points) Proof: If a function is analytic in the extended plane, then in particular, it's analytic at ∞. So it must have a definite finite value at ∞ and is continuous at ∞. Hence, it is bounded in a neighbourhood of ∞, say, $\{z : |z| > M\}$ for some positive number M. Meanwhile, this function is bounded in the closed disc $\{z : |z| \leq M\}$. So, this analytic function is bounded on the whole plane. By Liouville's Theorem, it has to be a constant. $\square$

6. (6 points) Solution:
(i) $\frac{e^z}{z^5} = \sum_{n=-5}^{\infty} \frac{z^n}{(n+5)!}$. So $a_n = \frac{1}{(n+5)!}$ and the principle part is $\sum_{n=-5}^{-1} \frac{z^n}{(n+5)!}$.
(ii) $\frac{\sin z}{(z-2\pi)^2} = \sum_{n=-1}^{\infty} \frac{(-1)^{n+1}+1}{2} \frac{(z-2\pi)^n}{(n+2)!}(-1)^{\frac{n+1}{2}}$. So $b_n = \frac{(-1)^{n+1}+1}{2(n+2)!}(-1)^{\frac{n+1}{2}}$ and the principle part is $(z - 2\pi)^{-1}$.
(iii) $\frac{z^6}{(1-z)^3} = -\frac{[(z-1)+1]^6}{(z-1)^3} = -\sum_{n=-3}^{3} C_6^{n+3}(z-1)^n$. So $c_n = -C_6^{n+3}$ and principle part is $-\sum_{n=-3}^{-1} C_6^{n+3}(z-1)^n$. $\square$

Section 3.9

4. (4 points) Proof: Since $f(z)$ is analytic for $|z| \neq 0$, we apply Theorem 9.2 to the case $\alpha = 0$ and get

$$\begin{aligned}
J_n(\omega) &= \frac{1}{2\pi i}\int_{|z|=1} \frac{e^{\omega(z-1/z)/2}}{z^{n+1}} dz \\
&= \frac{1}{2\pi i}\int_0^{2\pi} e^{\omega(e^{i\theta}-e^{-i\theta})/2} e^{-(n+1)i\theta} i e^{i\theta} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i\omega\sin\theta} e^{-in\theta} d\theta \\
&= \frac{1}{2\pi}[\int_0^{2\pi} \cos(\omega\sin\theta - n\theta)d\theta + \int_0^{2\pi} i\sin(\omega\sin\theta - n\theta)d\theta] \\
&= \frac{1}{\pi}\int_0^{\pi} \cos(\omega\sin\theta - n\theta)d\theta
\end{aligned}$$

$\square$

5. (4 points) Proof:

$$\frac{d^m}{d\omega^m} J_n(\omega) = \begin{cases} \frac{1}{\pi}\int_0^\pi \cos(\omega\sin\theta - n\theta)\sin^m\theta d\theta, & m = 0\text{mod}4 \\ \frac{1}{\pi}\int_0^\pi -\sin(\omega\sin\theta - n\theta)\sin^m\theta d\theta, & m = 1\text{mod}4 \\ \frac{1}{\pi}\int_0^\pi -\cos(\omega\sin\theta - n\theta)\sin^m\theta d\theta, & m = 2\text{mod}4 \\ \frac{1}{\pi}\int_0^\pi \sin(\omega\sin\theta - n\theta)\sin^m\theta d\theta, & m = 3\text{mod}4 \end{cases}$$

So

$$\left|\frac{d^m}{d\omega^m} J_n(\omega)|_{\omega=0}\right| = \begin{cases} \frac{1}{\pi}|\int_0^\pi \cos n\theta \sin^m\theta d\theta|, & m = 0\text{mod}2 \\ \frac{1}{\pi}|\int_0^\pi \sin n\theta \sin^m\theta d\theta|, & m = 1\text{mod}2 \end{cases}$$

By Problem 2, we see this is 0 for $0 \leq m < n$. So J_n has a zero of order n at $\omega = 0$ $\square$

Additional Problems on Chapter 3

3.3 (4 points) Proof: Assume $\max_{|z|=1} |1/z - f(z)| < 1$. Then

$$\left|\int_{|z|=1} [1/z - f(z)]dz\right| \leq \int_{|z|=1} |1/z - f(z)||dz| < 2\pi$$

By Cauchy's Theorem, $\int_{|z|=1}[1/z - f(z)]dz = 2\pi i$. Contradiction. $\square$

4.2 (4 points) Proof:

$$\frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}dz = \frac{1}{2\pi i}\int_C \frac{\sum_{k=1}^n \frac{P(z)}{z-z_k}}{P(z)}dz = \sum_{k=1}^n \frac{1}{2\pi i}\int_C \frac{dz}{z-z_k} = \sum_{k=1}^n N(z_k)$$

where $N(z_k)$ is the winding number of z_k with respect to C. $\square$