MATH 418 Function Theory
Homework 8 Solution

Due April 3

Section 4.2

4. (4 points) Solution:
(a)

$$\int_C \frac{2dz}{z^2+4iz-1} = \int_0^{2\pi} \frac{2ie^{i\theta}d\theta}{e^{2i\theta}+4ie^{i\theta}-1} = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$$

$z^2+4iz-1$ has two roots $-2+\sqrt{3}i$ and $-2-\sqrt{3}i$. Only $(\sqrt{3}-2)i$ is inside the unit disc. By (2.4), $Resf((\sqrt{3}-2)i) = 1/\sqrt{3}i$. So $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = 2\pi/\sqrt{3}$. $\square$
(b)

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} &= \int_0^{2\pi} \frac{d\theta}{2+\cos(\theta-\pi/2)} \\
&= \int_0^{3\pi/2} \frac{d\alpha}{2+\cos\alpha} + \int_{-\pi/2}^{0} \frac{d\alpha}{2+\cos(\alpha+2\pi)} \\
&= \int_0^{2\pi} \frac{d\alpha}{2+\cos\alpha} \\
&= 2\pi/\sqrt{3}
\end{aligned}$$

where the last step is by (2.7). $\square$

6. (4 points) Proof: Just follow the hint. First of all, LHS of the equality in the hint is $2\pi i \times Res(f;i) = 2\pi i \times 1/(2i) = \pi$. The first term of RHS tends to $\int_\infty^\infty \frac{dx}{1+x^2}$ as R goes to ∞. To see the second term goes to 0 as R goes to ∞, we note

$$\left|\int_{c_R} \frac{dz}{1+z^2}\right| \leq \int_0^\pi \left|\frac{iRe^{i\theta}}{1+R^2e^{2i\theta}}\right| d\theta \leq \int_0^\pi \frac{R}{R^2-1}d\theta \leq \pi\frac{R}{R^2-1} \longrightarrow 0$$

as R goes to ∞. $\square$

Section 4.3

1. (5 points) Remark: The solutions here are omitted since it's sort of tedious computation. And the contours can always be chosen as $\{z : z \in \mathbb{R}, -R < z < R\} \cup \{z : z = Re^{i\theta}, 0 \leq \theta \leq \pi\}$. The answers for the integrals are $\pi/\sqrt{2}, 2\pi/3, \pi/3, \pi/2, \pi/2$. $\square$

3. (5 points) Proof: As usual, we take the contour C as $\{z : z \in \mathbb{R}, -R < z < R\} \cup \{z : z = Re^{i\theta}, 0 \leq \theta \leq \pi\}$, and the function $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$. $f(z)$ has singularities $ai, -ai, bi$, and bi. Since the real parts of a and b are both positive, the singularities falling into C are ai and bi. So,

$$\begin{aligned} \int_C f(z)dz &= 2\pi i \left[\frac{e^{iz}}{(z^2+a^2)(z+bi)}\bigg|_{z=bi} + \frac{e^{iz}}{(z^2+b^2)(z+ai)}\bigg|_{z=ai} \right] \\ &= \frac{\pi}{a^2-b^2}(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}) \end{aligned}$$

We define $\{z : z \in \mathbb{R}, -R < z < R\}$ as II, and $\{z : z = Re^{i\theta}, 0 \leq \theta \leq \pi\}$ as I. And note

$$\left| \int_I f(z)dz \right| = \left| \int_0^\pi \frac{e^{i(R\cos\theta + i\sin\theta)}}{(R^2e^{2\theta i}+a^2)(R^2e^{2\theta i}+b^2)} iRe^{i\theta}d\theta \right|$$

$$\leq \int_0^\pi \frac{Re^{-R\sin\theta}}{(R^2-|a|^2)(R^2-|b|^2)}d\theta \leq \pi \frac{R}{(R^2-|a|^2)(R^2-|b|^2)}$$

It's clear that the last term goes to 0 as R goes to $+\infty$. So, we finally get

$$\frac{\pi}{a^2-b^2}(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}) = \int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = \int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$$

The last "=" is because $\frac{\sin x}{(x^2+a^2)(x^2+b^2)}$ is an odd function and it vanishes under the integration over the whole real line. $\square$

6. (9 points)

(a) If a and b are unequal complex numbers with positive real parts, prove

$$\int_{-\infty}^{\infty} \frac{x \sin x dx}{(x^2+a^2)(x^2+b^2)} = \pi \frac{e^{-a} - e^{-b}}{b^2 - a^2}$$

Proof: Let C_R be the semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\pi = C_R \cup [-R, R]$. Then $(x^2+a^2)(x^2+b^2) = 0$ if and only $x = ai$, $-ai$, bi, or $-bi$. Since Rea, $Reb > 0$, we have $Im(ai)$, $Im(bi) > 0$. So only ai, bi fall inside C when R is large enough. Hence

$$\int_C \frac{ze^{iz}dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i[Resf(ai) + Resf(bi)] = \pi i \frac{e^{-a} - e^{-b}}{b^2 - a^2}$$

Meanwhile

$$\left|\int_{C_R} \frac{ze^{iz}dz}{(z^2+a^2)(z^2+b^2)}\right| \le \int_0^\pi \frac{Rd\theta}{(R^2-|a|^2)(R^2-|b|^2)} \to 0$$

as $R \to \infty$. So let $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{xe^{ix}dx}{(x^2+a^2)(x^2+b^2)} = \pi i \frac{e^{-a}-e^{-b}}{b^2-a^2}$$

Equating the imaginary part, we get

$$\int_{-\infty}^{\infty} \frac{x\sin x dx}{(x^2+a^2)(x^2+b^2)} = \pi \frac{e^{-a}-e^{-b}}{b^2-a^2}$$

□

(b) By l'Hospital's rule, find the limit of the right-hand member as $b \to a$ in part (a). Then determine whether this limit agrees with the value of the integral for $b = a$.
Solution: By the same method as in part (a), we can find

$$\int_{-\infty}^{\infty} \frac{x\sin x dx}{(x^2+a^2)^2} = \frac{\pi e^{-a}}{2a}$$

This is exactly the limit of $\frac{\pi(e^{-a}-e^{-b})}{b^2-a^2}$ as $b \to a$, by l'Hospital rule. □

(c) If f denotes the integrand in part (a), and I denotes the value of the integral, show that

$$\left|\int_{-R}^{R} f(x)dx - I\right| \le \frac{\pi R}{(R^2-|a|^2)(R^2-|b|^2)}$$

where $R > \max(|a|,|b|)$.
Proof:

$$\begin{aligned}\left|\int_{-R}^{R} f(x)dx - I\right| &\le \left|\int_C f(x)dx - I\right| + \int_{C_R}\left|\frac{ze^{iz}}{(z^2+a^2)(z^2+b^2)}\right||dz| \\ &\le \frac{R\pi}{(R^2-|a|^2)(R^2-|b|^2)}\end{aligned}$$

□

Section 4.4

1. (3 points) Proof: Set $R = e^t$. Then

$$\frac{(\log R)^m}{R} = \frac{t^m}{e^t} < \frac{(m+1)!}{t} \to 0$$

as $t \to \infty$. So $\lim_{R\to\infty} \frac{(\log R)^m}{R} = 0$ along $R = e^t$. $\forall m \in \{0, 1, 2, 3, \cdots\}$, and set $\rho = R^{-1}$, then $|\rho(\log \rho)^m| = \frac{(\log R)^m}{R}$. So $\lim_{\rho\to 0+} |\rho(\log \rho)^m| = \lim_{R\to\infty} \frac{(\log R)^m}{R} = 0$. $\square$

Additional Problems on Chapter 4

4.1 (4 points) Proof: In $\{|z| \le r\}$, $f(z)$ can be written as $\prod_{i=1}^n (z - a_i)h(z)$ where h is analytic in $\{|z| \le r\}$. This shows g is analytic in $\{|z| \le r\}$ except for removable singularities. Since $r^2/\bar{a}_i$ is outside $\{|z| = r\}$ $(i = 1, \cdots, n)$ and $h(z) \neq 0$ in $\{|z| < r\}$, $g(z)$ does not vanish in $\{|z| < r\}$. Finally $|g(z)| = |f(z)|$ on $\{|z| = r\}$, since on $|z| = r$, $\left|\frac{r^2 - \bar{a}_i z}{r(t - a_i)}\right| = 1$ by Chapter 1 problem 2.1, and $|a_i/r| < 1$, $|z/r| = 1$. $\square$

4.2 (4 points) Proof: $|g(0)| = |r^n f(0)/\prod_{i=1}^n a_i|$. Meanwhile by maximum principle, $|g(0)| \le \max_{|z|=r} |g(z)| = \max_{|z|=r} |f(z)| = M(r)$. So we get $r^n/|a_1 \cdots a_n| \le M(r)/|f(0)|$. $\square$

Section 4.5

1. (6 points) Solution: Let $D = \mathbb{C}\backslash[-\infty, \infty]$, then D is an analytic branch for z^{a-1}. We use the same notation as in example 5.1. Then

$$I = \int_C \frac{z^{a-1}}{1 - z} dz = -2\pi i$$

Furthermore,

$$I = \int_R^\varepsilon \frac{(re^{i\pi})^{a-1}}{1 + r} e^{i\pi} dr + \int_\varepsilon^R \frac{(re^{-i\pi})^{a-1}}{1 + r} e^{-i\pi} dr + J_1 + J_2$$

In D, $z^{a-1} = e^{(a-1)(\log|z| + iargz)}$ where $-\pi < argz < \pi$. So the first integral $= e^{i\pi} e^{(a-1)(\log r + \pi i)}/(1 + r) = e^{(p-1)\log r - \pi q} e^{(q \log r + \pi p)i}/(1 + r)$. We do similar thing to the second integral and get

$$I = \int_\varepsilon^R \frac{r^{p-1} e^{iq\log r}(-e^{-\pi q + \pi p i} + e^{\pi q - \pi p i})}{1 + r} dr + J_1 + J_2$$

On the circle of radius R and ε, respectively,

$$\left|\frac{z^{a-1}}{1 - z}\right| \le \frac{R^{p-1} e^{\pi|q|}}{R - 1}, \left|\frac{z^{a-1}}{1 - z}\right| \le \frac{\varepsilon^{p-1} e^{\pi|q|}}{1 - \varepsilon}$$

Hence $|J_1| \le \frac{2\pi R e^{\pi|q|}}{R-1}$, $|J_2| \le \frac{2\pi \varepsilon^p e^{\pi|\varepsilon|}}{1-\varepsilon}$. Since $0 < p < 1$, by letting $R \to \infty$ and $\varepsilon \to 0$, we get

$$\int_0^\infty \frac{r^{p-1} e^{iq\log r}}{1 + r} (e^{\pi q - \pi p i} - e^{-\pi q + \pi p i}) dr = -2\pi i$$

By $\sin\pi(p+iq) = \sin\pi p\cosh\pi q + i\cos\pi p\sinh\pi q$,

$$\int_0^\infty \frac{r^{p-1}}{1+r}e^{iq\log r}dr = \frac{2\pi i}{e^{-\pi q+\pi pi} - e^{\pi q-\pi pi}} = \frac{\pi}{\sin\pi(p+iq)} = \frac{\pi}{\sin\pi p\cosh\pi q + i\cos\pi p\sinh\pi q}$$

Equating the real and imaginary parts, we get

$$\int_0^\infty \frac{t^{p-1}}{t+1}\cos(q\log t)dt = \frac{\pi\sin\pi p\cosh\pi q}{(\sin\pi p\cosh\pi q)^2 + (\cos\pi p\sinh\pi q)^2}$$

$$\int_0^\infty \frac{t^{p-1}}{t+1}\sin(q\log t)dt = \frac{-\pi\cos\pi p\sinh\pi q}{(\sin\pi p\cosh\pi q)^2 + (\cos\pi p\sinh\pi q)^2}$$

$\square$

5. (6 points) Proof: Let $f(z) = \sqrt{z}\log z/(1+z)^2$. Let $D = \mathbb{C}\backslash[0,\infty]$. Then D is an analytic branch of $f(z)$. We let C be the contour enclosed by circles of radius R and radius ε. Imitating example 5.1, we cut C into two parts, one part contains -1, and the other one doesn't. Then $\int_C f(z)dz = 2\pi i Resf(-1)$. $Resf(-1) = (z^{1/2}\log z)'|_{z=-1} = (z^{1/2}/z + z^{1/2}\log z/2z)|_{z=-1} = \pi/2 - i$. So

$$\pi^2 i + 2\pi = \int_\varepsilon^R \frac{\sqrt{x}\log x}{(1+x)^2}dx - \int_\varepsilon^R \frac{(xe^{2\pi i})^{1/2}\log(xe^{2\pi i})}{(1+x)^2}dx + J_1 + J_2$$

where J_1 and J_2 are the integration of $f(z)$ along the circle of radius R and radius ε, respectively. So

$$\pi^2 i + 2\pi = \int_\varepsilon^R \frac{\sqrt{x}\log x + \sqrt{x}(\log x + 2\pi i)}{(1+x)^2}dx + J_1 + J_2$$

On $|z| = R$, $|f(z)| \leq \sqrt{R}(R+2\pi)/(R-1)^2$ and on $|z| = \varepsilon$, $|f(z) \leq \sqrt{\varepsilon}(\log\varepsilon + 2\pi)/(1-\varepsilon)^2$. So let $R \to \infty$ and $\varepsilon \to 0$, we get $|J_1| \to 0$ and $|J_2| \to 0$, respectively. Hence $\pi^2 i + 2\pi = 2\int_0^\infty \frac{\sqrt{x}\log x}{(1+x)^2}dx + 2\pi i\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}dx$. Equating real parts and imaginary parts, we're done. $\square$