MATH 418 Function Theory
Homework 9 Solution

Due April 10

Section 4.6

1. (4 points) Proof: Let $f = 2z^5$, $g = 8z - 1$ and $C = \{|z| = 2\}$. Then on C, $|f(z)| = 64 > 17 \geq |g(z)|$. By Rouché's Theorem, $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C. Because f has 5 roots inside C (0 with multiplicity of 5), so does $f(z) + g(z) = 2z^5 + 8z - 1$. □

2. (4 points) Proof: Let $f(z) = 8z - 1$, $g(z) = 2z^5$ and $C = \{|z| = 1\}$. Then on C, $f(z) \geq 7 > 2 = g(z)$. So $f(z)$ and $g(z) + f(z)$ have the same number of zeros inside C. Hence $2z^5 + 8z - 1 = f(z) + g(z)$ has only one root in $\{|z| < 1\}$. Meanwhile the real polynomial $2x^5 + 8x - 1$ has opposite signs at $x = 0$ and $x = 1$. So it must have a positive real root in $(0, 1)$. This shows the complex polynomial $2z^5 + 8z - 1$ has just one root in $\{|z| < 1\}$, and this root is real and positive. □

6. (4 points) Proof: Let $f(z) = -2z$, $g(z) = e^z - 1$, and $C = \{|z| = 1\}$. Then on C, $|f(z)| = 2 > e - 1$, and $|g(z)| = |\int_0^z e^\zeta d\zeta| \leq e - 1$. So f and $f + g$ have the same number of roots in $\{|z| < 1\}$, i.e. only one root. □

7. (4 points) Proof: The fundamental theorem of algebra can be stated as: any complex polynomial has at least one root. Let $f(z) = a_n z^n$, $g = a_{n-1}z^{n-1} + \cdots + a_0$ and $C = \{|z| = R\}$. For R large enough, we have $|f(z)| = |a_n|R^n > \sum_{k=0}^{n-1} |a_k|R^k \geq |g(z)|$. So $f(z)$ and $f(z)+g(z)$ have the same number of roots inside C. Since f has n roots inside C, so does $f + g$. We're therefore done. □

8. (4 points) Proof: Assume there exists z_0 inside C such that $|g(z_0)| > m$. Let $f(z) = -g(z_0)$, then on C, $|f(z)| > m \geq |g(z)|$. So $-g(z_0)$ and $-g(z_0) + g(z)$ have the same number of zeros inside C, i.e. 0. Contradicting with the fact z_0 is a zero of $-g(z_0) + g(z)$ inside C. □

10. (4 points) Proof: WLOG, we assume $P(z) = \prod_{i=1}^n (z - a_i)$ with $a_i \neq 0$, $i = 1, \cdots, n$. Then $P'(z)/P(z) = \sum_{i=1}^n 1/(z - a_i)$. So $\sum_{i=1}^n 1/a_i = -P'(0)/P(0)$. Meanwhile $(\frac{P'(z)}{P(z)})' = \sum_{i=1}^n \frac{-1}{(z-a_i)^2}$. So

$$\sum_{i=1}^{n} \frac{-1}{(z - a_i)^2} = \frac{P"(z)P(z) - (P'(z))^2}{P(z)^2}$$

Let $z = 0$, we then get the desired equality. $\square$

Section 4.7

1. (4 points) Proof: Apply the Poisson formula to e^{-z}, we get

$$e^{-z} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{|z - i\omega|^2} e^{i\omega} d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2 + (y-\omega)^2} e^{i\omega} d\omega$$

Equating the real and imaginary parts, we get

$$e^{-x}\cos y = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2 + (y-\omega)^2} \cos\omega d\omega$$

$$e^{-x}\sin y = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2 + (y-\omega)^2} \sin\omega d\omega$$

$\square$

4. (10 points) Solution:

(i)

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds$$

We let $r > 0$, $C_R = \{|z| = R, Rez \le c\}$, and $C = C_R \cup [c - ri, c + ri]$. Then

$$\begin{aligned}
& \frac{1}{2\pi i}\int_{c-ir}^{c+ir} e^{st}F(s)ds + \frac{1}{2\pi i}\int_{C_R} \frac{ae^{st}}{s^2 + a^2} ds \\
= & \frac{1}{2\pi i}\int_{C} \frac{ae^{st}}{s^2 + a^2} ds \\
= & Res(\frac{ae^{st}}{s^2 + a^2}; ai) + Res(\frac{ae^{st}}{s^2 + a^2}; -ai) \\
= & \frac{ae^{ati}}{2ai} + \frac{ae^{-ati}}{-2ai} \\
= & \sin at
\end{aligned}$$

On C_R, $|ae^{st}/(s^2 + a^2)| \le \frac{|a|e^{|t|c}}{R^2 - a^2}$ since $Rez \le c$ on C_R. So

$$\left|\int_{C_R} \frac{ae^{st}}{s^2 + a^2} ds\right| \le \frac{|a|e^{|t|c} 2\pi R}{R^2 - a^2} \to 0$$

as $R \to \infty$. Hence $f(t) = \sin at$ by letting $r \to \infty$. □

(ii) We use the same contour as in (i)

$$\begin{aligned} & \frac{1}{2\pi i}\int_{c-ir}^{c+ir}\frac{ae^{st}}{s(s^2+a^2)}ds + \frac{1}{2\pi i}\int_{C_R}\frac{ae^{st}}{s(s^2+a^2)}ds \\ = & \frac{1}{2\pi i}\int_C \frac{ae^{st}}{s(s^2+a^2)}ds \\ = & \frac{a}{a^2} + \frac{ae^{ati}}{ai(2ai)} + \frac{ae^{-ati}}{(-ai)(-2ai)} \\ = & \frac{1}{a} - \frac{1}{a}\cos at \end{aligned}$$

On C_R, $|\frac{ae^{st}}{s(s^2+a^2)}| \le \frac{ae^{|t|c}}{R(R^2-a^2)}$. So by letting $R \to \infty$ first and then $r \to \infty$, we get $f(t) = (1-\cos at)/a$. □

By similar methods, we have

(iii) te^{-at}. □

(iv) $t^7/7!$. □

(v) $\cos at$. □

Section 4.8

1. (3 points) Proof: It's obvious that $f(z)$ is analytic at $z = \infty$. By formula (8.1)

$$Res(f;\infty) = -\frac{1}{2\pi i}\int_{\{|z|=R\}}\frac{dz}{z} = -\frac{1}{2\pi i}\int_0^{2\pi}\frac{iRe^{i\theta}d\theta}{Re^{i\theta}} = -\frac{1}{2\pi}\times 2\pi = -1$$

□

6. (9 points) Solution:

(i) Use the contour C in Figure 8.3. And we get

$$I = \int_C \frac{z^2dz}{(1+z^2)\sqrt{1-z^2}} = 2\int_{-1+\eta}^{1-\eta}\frac{z^2dz}{(1+z^2)\sqrt{1-z^2}} + J_1 + J_2$$

where J_1 and J_2 are the integrals over the small circles of radius ε and η, respectively. By Theorem 8.2

$$I = 2\pi i[Res(f;i) + Res(f;-i) + Res(f;\infty)] = \pi(2-\sqrt{2})$$

And we have, similar to example 8.2, that as $\varepsilon \to 0$, $\eta \to 0$, J_1, $J_2 \to 0$. So

$$\int_0^1 \frac{x^2dx}{(1+x^2)\sqrt{1-x^2}} = \frac{1}{2}\int_{-1}^1\frac{x^2dx}{(1+x^2)\sqrt{1-x^2}} = \frac{\pi}{4}(2-\sqrt{2})$$

□

(ii) We use the contour in Figure 8.4, and we get $35\pi/128$. □

(iii) Choose the contour in Fig. 8.3, and we get $\frac{\pi}{4}(\sqrt{2}-1)$. □