Mathematics 418: Functions of One Complex Variable

Spring 2003

Notes on Riemann Surfaces

Riemann surfaces are a special case of a more general mathematical object called a manifold. Ignoring some technical complications, we define manifolds in the following way:

A C^r n-dimensional *manifold* M consists of a set (again denoted by M) together with a collection of subsets U_i such that

- The set M is the union of the U_i.
- There are $1-1$ maps $\phi_i : U_i \to R^n$ with image a domain.
- If $U_i \cap U_j$ is nonempty, then $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is r -times differentiable and its Jacobian is always nonsingular.

The collection of U_i and ϕ_i is called an *atlas* for M and each ϕ_i is a *chart* on U_i. The $\phi_j \circ \phi_i^{-1}$ are *transition functions.*

A *Riemann surface* is a two dimensional manifold for which the charts are complex analytic functions of C. Any domain of C with the identity map is a Riemann surface.

Example: Another example of a Riemann surface is the extended complex plane that we can regard as a two dimensional sphere S^2. We take two charts $U_1 = S^2$ - {north pole} and $U_2 = S^2$ - {south pole}. The maps ϕ_1 and ϕ_2 are defined so that $\phi_1(U_1)$ and $\phi_2(U_2)$ both map onto C and the map $\phi_2 \circ \phi_1^{-1}(z) = 1/z$.

There are two common ways of constructing Riemann surfaces: *analytic continuation* and as solutions of systems of equations. One approach to analytic continuation is to consider Taylor series expansions of an analytic function. If f is analytic in the disk $B_r(\alpha)$ of radius r centered at the point α, then the Cauchy integral formula and Taylor's theorem imply that the Taylor series of f converges in $B_r(\alpha)$. We can use the Taylor series to define the function f: start with f being defined by its Taylor series exapnsion in $B_r(\alpha)$ with r chosen as the radius of convergence of the series. Given a point $\beta \in B_r(\alpha)$, we can compute the Taylor series at β and determine its radius of convergence s. This will define the analytic function in $B_s(\beta)$. Now $B_s(\beta)$ may well contain points not in $B_r(\alpha)$, so we have extended the domain of f. Since

both series give the same function values on $B_r(\alpha) \cap B_s(\beta)$, f is the same function we started with.

Example: Let $f(z) = 1/z$ in $C - \{0\}$. At the point α we have

$$\frac{1}{z} = \frac{1}{\alpha + (z-\alpha)} = \frac{1}{\alpha}(\frac{1}{1-(1-z/\alpha)}) = \frac{1}{\alpha}\sum_{i=0}^{\infty}(1-z/\alpha)^i = \sum_{i=0}^{\infty}(-\alpha)^{-i-1}(z-\alpha)^i$$

This has radius of convergence α. We can cover $C - \{0\}$ by disks of the form $B_\alpha(\alpha)$. The same analysis can be applied to $g(z) = \log(z) = \int f(z)dz$ by integrating the Taylor series of f. However, we encounter the same problem that we have seen before in defining $\log(z)$, namely that when we take values from a sequence of disks that follows a closed path with non-zero winding number around the origin, then the value of $\log(z)$ we obtain is different from the original value. This prompts us to construct a Riemann surface on which the closed curve no longer returns to the same point.

The difficulties in defining single valued functions by analytic continuation and by integration along paths are similar: in both cases the difficulties vanish in simply connected domains. Topology can be used to transcend the problem with a construction that produces the *universal covering space* of a manifold. If M and P are differentiable manifolds, then a map $\psi : P \to M$ is a *covering map* if ψ is a *local diffeomorphism* (i.e. $D\psi$ is invertible at every point of P) and onto. An important example

Example: Consider $M = S^1 \subset R^2$ to be the unit circle and $P = R^1$. The map $\psi : P \to M$ defined by $\psi(x) = \exp(2\pi i x)$ is a covering map.

Two closed curves $\gamma_0, \gamma_1 : [0,1] \to M$ on the manifold M are *homotopic* if there is a continuous map is simply connected if there is a continuous map $\Gamma : [0,1] \times [0,1] \to M$ such that $\Gamma(0,t) = \gamma_0(t), \Gamma(1,t) = \gamma_1(t), \Gamma(s,0) = \Gamma(s,1)$. A connected manifold M is simply connected if every closed curve is homotopic to a constant curve. A covering map $\psi : P \to M$ of a connected manifold M is a *universal cover* if P is simply connected.

Every connected manifold M has a universal cover P that is constructed from curves on M. Pick a *base point* $c \in M$ and consider curves $\gamma_0 : [0,1] \to M$ with $\gamma(0) = c$. The covering space P is defined as a quotient of the space of curves of M (beginning at c) by an equivalence relation. Given two curves $\gamma_0, \gamma_1 : [0,1] \to M$ with $\gamma_0(0) = \gamma_1(0) = c$ and $\gamma_0(1) = \gamma_1(1)$, we can construct a closed curve $\gamma_{01} : [0,1] \to M$ defined by $\gamma_{01} = \gamma_0(2t)$ for $t \in [0, 1/2]$ and $\gamma_{01} = \gamma_1(2(1-t))$ for $t \in [1/2, 1]$. If γ_{01} is homotopic to a constant closed curve, then we regard γ_0 and γ_1 as equivalent. The covering map $\psi : P \to M$ is defined by $\psi(\gamma) = \gamma(1)$, its endpoint. There is a substantial amount of straightforward argument required to prove that P has the

structure of a manifold and that it is simply connected. Moreover, $\psi^{-1}(c)$ has a group structure defined by concatenation of loops, with the group operation defined as concatenation of loops. Inverses are obtained by reversing the orientation along a loop. Any two universal covers P_1, P_2 of M are diffeomorphic - there is a 1-1 smooth map $\phi : P_1 \to P_2$ with a smooth inverse. Details of this construction are described in topology couses.

If M is a Riemann surface, then its universal cover P is also a Riemann surface. In terms of the construction described above, concatenation of curves can be used to define coordinates on P in terms of coordinates on M. Analytic continuation of a function f on M along homotopic curves always produces the same value. Therefore analytic continuation of a function f defined on all of M produces a single valued analytic function on P. Similarly, path integration of an analytic function defined on M produces a single valued analytic function on P.

Example: The map $\exp : C \to C - \{0\}$ is a universal cover of $C - \{0\}$.

A central question in complex analysis is the classification of Riemann surfaces. Given two Riemann surfaces M_1 and M_2, we say they are *isomorphic* if there is an analytic map $f : M_1 \to M_2$ that is 1-1, onto and has non-zero derivative. This implies that M_1 and M_2 are homeomorphic as topological spaces, but there are homeomorphic Riemnann surfaces that are not isomorphic. Given a Riemann surface M, we would like to parameterize the equivalence classes of Riemann surfaces homeomorphic to M, the equivalence class being isomorphism. There are two cases of particular interest in which the answers are particularly elegant: M simply connected and M compact. For the case of simply connected M, we have the following.

Example: The complex plane C and the unit disk $D = \{z| \, |z| < 1\}$ are homeomorphic, but not isomorphic. Proof: An analytic function $f : C \to D$ is a bounded function on all of C. Liouville's states that f is constant.

Theorem: Every simply connected Riemann surface is isomorphic to C, the extended complex plane S^2 or $H = \{z| \, |z| < 1\}$.

The proof of this theorem is complicated - the hardest part being the *Riemann Mapping Theorem* that we shall discuss in detail over several lectures. We shall also study geometry associated to the simply connected Riemann surfaces as well. In each case, the surfaces have a group of analytic self-mappings that is quite large and gives a way of relating complex analysis to classical Euclidean and non-Euclidean geometry.

A second common way of defining Riemann surfaces is as the solutions to n

analytic equations in $n+1$ complex variables. When the Jacobian of the equations has full rank, then the implicit function theorem says that the solutions will form a one (complex) dimensional manifold.

Example: Let $P(z)$ be a polynomial of degree $2g+2$ with simple roots. Let M be the set of solutions of the equation $w^2 = P(z)$. The Jacobian of this equation is $(w, P'(z))$. When $w = 0$ on M, $P'(z) \neq 0$ since P has simple roots. Thus the Jacobian always has rank 1. There is a natural way to extend the set of solutions at infinity so that the extended M becomes a compact Riemann surface. We can regard the Riemann surface as a two sheeted cover of the extended complex plane by projecting M onto the z coordinate. There are branch points with behavior like the Riemann surface of the function z^2 at each of the roots of P. By cutting M along disjoint curves that join pairs of the branch points, we can examine its topology. It turns out to be homeomorphic to a sphere with g handles attached. The number g is called the *genus* of this compact Riemann surface. Compact Riemann surfaces of genus 1 are called *elliptic curves* and are topologically a two dimensional torus. The universal cover of an elliptic curve is the complex plane (not the disk!) and the fundamental domain can be chosen to be a parallelogram. Riemann surfaces of genus $g > 1$ have universal cover the disk (not the plane!) and the fundamental domain can be chosen to be a non-Euclidean polygon of $4g$ sides in the disk.