

# Mathematics 418: Functions of One Complex Variable

Spring 2003

## Notes on Riemann Surfaces - Installment 2

We observed that the solutions of the equation  $w^2 = P(z)$  form a smooth Riemann surface  $M$  if  $P$  has simple roots. Here we discuss the surface further. We would like to construct an explicit covering map  $\pi$  from the universal cover of  $M$  to  $M$ . We do this by defining  $\pi^{-1}$  as an integral rather than by constructing  $\pi$  itself. The simplest case for this procedure is the construction of the function  $\sin$  via

$$\sin^{-1}(u) = \int_0^u \frac{dz}{\sqrt{1-z^2}}$$

. In calculus, this definition is used with  $z \in [-1, 1]$ , so the positive square root can be selected without ambiguity. To use this definition in the whole complex plane, this becomes an issue that can be addressed by interpreting the integral as an integral on the Riemann surface  $w^2 + z^2 = 1$ .

To define integration on Riemann surfaces, we need to rely upon the substitution formula for integration. Since different coordinates can be used to describe pieces of a Riemann surface, the value of a path integral that we obtain should be independent of the coordinate systems used along the path. For this to make sense, integrands  $f(z)dz$  must be thought of as a differential forms or differentials that transform under coordinate change in the manner described by the substitution formula:  $f(z)dz = f \circ h(w)h'(w)dw$  if  $z = h(w)$  is a coordinate change. The factor  $h'(w)$  in this formula distinguishes the integrand from a function. Functions satisfy the simpler formula  $g(z) = g \circ h(w)$ .

On the Riemann surface  $w^2 = P(z)$ , we can use  $z$  as a local coordinate except at the branch points where  $P(z) = 0$ . There we can use  $w$  as a coordinate. The integrals we compute are

$$\int_0^u \frac{dz}{\sqrt{P(z)}} = \int_0^v \frac{2dw}{\sqrt{P(0)} P'(z(w))}$$

where  $v^2 = P(u)$ . When the degree of  $P$  is larger than 2, using  $w$  as a local coordinate is awkward since we need to solve for  $z$  as a function of  $w$  on the surface. Thus we work with the integrals expressed in terms of  $z$ , but writing the integrals in terms of  $w$  makes it clear that they converge at the branch points.

We also want to study the surface and the integrals at  $\infty$ . The “right” way to do this is to adjoin a “line at infinity” to  $C^2$ , making it into a *projective space*, but we don’t describe that construction here. Instead, we note that if the degree of  $P$  is even, then the branches of the square root are separated at  $\infty$  - i.e., if we traverse a curve that surrounds all of the roots of  $P$  in the  $z$  plane, then the value of  $\sqrt{P(z)}$  returns to its original value. Thus we extend  $M$  by adjoining a point at infinity to each of its two sheets, obtaining a compact surface when we do.

Let us now examine two examples.

**Example:**  $P(z) = 1 - z^2$

If we set  $\zeta = 1/z$ ,

$$\frac{dz}{\sqrt{1 - z^2}} = \frac{-d\zeta}{\zeta\sqrt{\zeta^2 - 1}}$$

which has a simple pole at the origin and residue  $\pm i$  depending upon which square root of  $-1$  is chosen. If we remove the two points at  $\infty$  from  $M$  once more, then the integral for  $\sin^{-1}(u)$  above is a multivalued function, with different values differing by multiples of  $2\pi$  (that’s  $2\pi i$  times the residue). The inverse function of  $\sin^{-1}(u)$  is well defined and is periodic with period  $2\pi$ . We have not established that the range of  $\sin^{-1}(u)$  is the whole complex plane, so that  $\sin$  is defined everywhere.

**Example:**  $P(z) = (1 - z^2)(1 - k^2 z^2)$

If we set  $\zeta = 1/z$ ,

$$\frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} = \frac{-d\zeta}{\sqrt{(\zeta^2 - 1)(\zeta^2 - k^2)}}$$

which is finite at the origin. So now

$$f(u) = \int_0^u \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}$$

is defined on the entire extended surface  $M$ , but it is still multivalued due to the fact that  $M$  is not simply connected. Let us come back to the topology of  $M$  to understand better what the image of  $f$  looks like. To make the description simpler, let us assume that  $0 < k < 1$ .

To obtain a fundamental domain for the image of  $f$ , we want to remove a (small) set  $S$  from  $M$  that leaves the remaining part  $U$  connected and simply connected. We will make the choice that  $S$  is the set of points of  $M$  that project onto  $[1, \infty) \cup \infty \cup (-\infty, -1/k]$  in the extended  $z$  plane. Then  $(z, w) = (-1, 0)$  is in  $U$ , but the other intersections of  $M$  with  $w = 0$  are not. (It is pretty easy to see that  $U$  looks a lot like the Riemann surface of  $w^2 = z$  with a pair of disjoint slits extending to  $\infty$ . This is simply connected. If one also removes the points that project onto the segment  $[-1/k, -1]$ , then the two sheets are separated.) The set  $S$  consists of two simple closed curves, one projecting onto the segment  $[1, 1/k]$  and one projecting onto the segment  $[1/k, -1/k]$  extending through  $\infty$ . These two closed curves intersect at the point  $(1/k, 0) \in M$ . If one views a torus by identifying opposite sides of a rectangle, then  $M$  has the same topological structure. The set  $S$  is homeomorphic to the figure eight obtained by identifying opposite sides of the boundary of a rectangle.

If we restrict the definition of  $f$  to curves that are inside  $U$  except perhaps at their endpoints, then its image will be a bounded set in the plane. There are curves approaching each boundary point of  $U$  that are not homotopic, and these give different values of  $f$ . To see that  $f$  is multivalued, consider the closed curve projecting onto the segment  $[-1, 1]$ . Integration over the “top half” of this curve (i.e. positive square root) gives

$$\int_{-1}^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

while integration over the bottom half gives

$$\int_1^{-1} \frac{dz}{-\sqrt{(1-z^2)(1-k^2z^2)}} = \int_{-1}^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

Since the integrands are positive on  $[-1, 1]$ , so is the integral. A similar argument shows that the integral over the closed curve projecting onto  $[1, 1/k]$  is pure imaginary.

We conclude that there are *two* periods of  $f^{-1}$  that are not multiples of one another. We say that  $f^{-1}$  is a doubly periodic or *elliptic* function - though it does have poles since  $f$  was defined at  $\infty$ . There is another way to reconstruct the torus from  $f^{-1}$ . Consider the translations of  $C$  by all integer combinations of the periods. These form a subgroup  $\mathcal{L}$  of the group of all translations  $\mathcal{T}$ . The torus can be described as the quotient  $\mathcal{T}/\mathcal{L}$ .

If we consider polynomials  $P$  of degree  $2g + 2 > 4$  with simple zeros, then the Riemann surface  $M$  defined by  $w^2 = P(z)$  can be studied in a simpler manner to the torus. However, the genus  $g$  (number of holes) in  $M$  is larger, and the universal covering space of  $M$  is isomorphic to the disk rather than all of  $C$ . The geometry of a map from the universal cover to  $M$  to  $M$  is determined by a subgroup of  $\text{sl}(2, R)$  and  $M$  can be reconstructed from a non-Euclidean polygon with  $4g$  sides by pairwise identifications of the sides.