MATH 6530 HOMEWORK 1

DUE SEPTEMBER 8TH

1. Exercises

(1) Show that a vector bundle $E \longrightarrow X$ has k independent sections if and only if it has a trivial k-dimensional subbundle.
(2) Prove that $G_n(\mathbf{R}^k)$ is homeomorphic to $G_{k-n}(\mathbf{R}^k)$.
(3) Prove explicitly that γ_{12} is the Mobius bundle.
(4) Check that $[X, Y]$ is a group if $Y \simeq \Omega Z$ and that it is abelian if $Z \simeq \Omega Z'$. Prove that the adjunction isomorphism $[\Sigma X, Y] \cong [X, \Omega Y]$ respects this group structure.

2. Problems

(1) Let $p\colon TS^n \longrightarrow S^n$ be the tangent bundle of S^n. Show that if p admits an everywhere-nonzero section then the identity map is homotopic to the antipodal map $x \mapsto -x$. When n is even, show that the antipodal map is homotopic to the map $(x_0, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$, which has degree -1. Use this to show that TS^n is not trivial. (This means that S^n is not *parallelizable*.)
(2) Let $p_1\colon E_1 \longrightarrow B$, $p_2\colon E_2 \longrightarrow B$ be two bundles over B. Prove that the bundle $E_1 \oplus E_2$ which has as it's fiber over b the vector space $p_1^{-1}(b) \oplus p_2^{-1}(b)$ is a vector bundle over B. Do the same for $E_1 \otimes E_2$,

Now suppose that we have a functor $\star\colon \mathbf{Vect} \times \mathbf{Vect} \longrightarrow \mathbf{Vect}$. What conditions are required on $\star$ to make it possible to define a vector bundle $E_1 \star E_2$? More generally, what if $\star$ has as its domain a subcategory of $\mathbf{Vect}^k$? (Examples you may want to work: dual bundle, quotient bundle, orthogonal complements of subbundles.)
(3) $\mathbf{R}^n$ has a lot of good structure on it. For example, it has the Euclidean metric. Show that if B is paracompact then there exists a Euclidean metric on E: a function $\mu\colon E \longrightarrow \mathbf{R}$ such that its restriction to each fiber is a positive definite quadratic form.