MATH 6530 HOMEWORK 3

DUE OCTOBER 27

(1) Show that the Steifel–Whitney classes of a Cartesian product are given by

$$w_k(E \times E') = \sum_{i=0}^{k} w_i(E) \times w_{k-i}(E') \in H^k(B \times B').$$

(2) Prove that if the n-dimensional manifold M can be immersed into $\mathbf{R}^{n+1}$ then each $w_i(TM)$ is equal to the i-fold cup product $w_1(TM)^i$. If $\mathbf{R}P^n$ can be immersed into $\mathbf{R}^{n+1}$ show that n must be of the form $2^r - 1$ or $2^r - 2$.

(3) An orientation for a vector bundle $E \longrightarrow B$ is a function which assigns an orientation for each fiber F of E such that for every point $b \in B$ there exists a neighborhood $U \ni b$ and a local trivialization $\varphi_b \colon p^{-1}(U) \longrightarrow U \times \mathbf{R}^n$ which is orientation-preserving on each fiber. A bundle is *orientable* if there exists an orientation on the bundle.

- Prove that an orientation on E gives a preferred generator $u \in H^n(D(F)/S(F); \mathbb{Z})$ for every fiber F. (For each F, $D(F)$ is the unit disk in F and $S(E)$ is the unit sphere.)
- Prove that the Möbius strip is not orientable.
- Prove that every complex vector bundle is orientable.
- Prove that for any n, $\gamma_n \oplus \gamma_n$ is orientable with $w_{2n}(\gamma_n \oplus \gamma_n) \neq 0$.

(4) Let $p \colon E \longrightarrow B$ be a rank-n bundle which is classified by a map $f \colon B \longrightarrow G_n$. Let $\widetilde{G}_n$ be the Grassmannian of oriented n-planes.
  (a) Prove that $\widetilde{G}_n \longrightarrow G_n$ is a 2-fold covering and that $\widetilde{G}_n$ is simply connected.
  (b) Prove that E is orientable if and only if f lifts to $\widetilde{G}_n$.
  (c) Note the natural isomorphisms

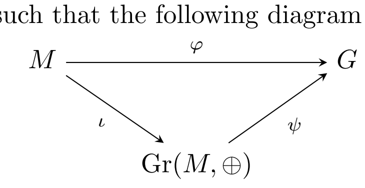
$$H^1(B; \mathbb{Z}/2) \longrightarrow \mathrm{Hom}(H_1(B), \mathbb{Z}/2) \longrightarrow \mathrm{Hom}(\pi_1(B), \mathbb{Z}/2).$$

  Use this to show that $f^*(w_1(\gamma_n))$ corresponds to the induced map $f_* \colon \pi_1(B) \longrightarrow \pi_1(G_n)$.
  (d) Conclude that E is orientable if and only if $w_1(E) = 0$.

(5) For which n does there exist an $n+1$-manifold B such that $\partial B \cong \mathbf{R}P^n$?

(6) Calculate the characteristic classes of the tangent bundle of the 2-torus $S^1 \times S^1$ in two different ways.

(7) Let M be an abelian monoid with operation $\oplus$, and define $\mathrm{Gr}(M, \oplus)$ be the free abelian group generated by the elements of M modulo the relation that whenever $m + m' = m''$ in M we have $[m] + [m'] = [m'']$ in $\mathrm{Gr}(M, \oplus)$. Write $\iota \colon M \longrightarrow \mathrm{Gr}(M, \oplus)$ for the map taking m to $[m]$.
  (a) Prove that ι is a monoid homomorphism.
  (b) Prove that $\mathrm{Gr}(M, \oplus)$ is the group completion of M. In other words, prove that for any monoid homomorphism $\varphi \colon M \longrightarrow G$ where G is an abelian group, there exists a unique group homomorphism $\psi \colon \mathrm{Gr}(M, \oplus) \longrightarrow G$ such that the following diagram commutes:



  (In other words, Gr is the left adjoint to the forgetful functor from abelian monoids to abelian groups.)
  (c) Give an example of a monoid M where ι is injective.
  (d) Give an example of a monoid M where ι is nonzero but not injective.
  (e) Give an example of a nontrivial monoid M where ι is zero.