# MATH 6530 HOMEWORK 4

DUE NOVEMBER 17

(1) (a) Prove that the ring $\Sigma$ of symmetric functions in countably many variables $t_1, t_2, \ldots$ is a $\lambda$-ring, where $\lambda^n e_1 = e_n$. (Here, $e_i$ is the $i$-th elementary symmetric function.)
    (b) Prove that $\Lambda(R)$ is a $\lambda$-ring for any ring $R$.
    (c) Let $T$ be any $\lambda$-ring. Prove that for any $x \in T$ there is a unique homomorphism of $\lambda$-rings $\Sigma \longrightarrow T$ sending $e_1$ to $x$. Where does the polynomial $\sum_{n \geq 0} t_n^k$ go?

(2) Use $K$-theory to prove that there is no retraction from the disk to the sphere.

(3) Given spaces $X$, $Y$, we can define an *external product* $*$ by

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \xrightarrow{pr_1^* \otimes pr_2^*} \widetilde{K}^0(X \times Y) \otimes \widetilde{K}^0(X \times Y) \xrightarrow{\text{mult}} \widetilde{K}^0(X \times Y).$$

    (a) Prove that there exists a relative version of the external product,

$$\widetilde{*} \colon \widetilde{K}^0(X/A) \otimes \widetilde{K}^0(Y/B) \longrightarrow \widetilde{K}^p(X \times Y/((A \times Y) \cup (X \times B))).$$

    (b) Prove that for any subspace $A$ of $X$ the diagram

$$\begin{array}{ccc} \widetilde{K}^0(X/A) \otimes \widetilde{K}^0(X/A) & \xrightarrow{\quad\otimes\quad} & \widetilde{K}^0(X/A) \\ & \searrow^{\widetilde{*}} & \uparrow{\scriptstyle \Delta^*} \\ & & \widetilde{K}^0(X \times X/((A \times X) \cup (X \times A))) \end{array}$$

    Here, $\Delta \colon X/A \longrightarrow X/A \times X/A$ is the diagonal map.
    (c) Prove that for any space $X$ which is the union of $n$ open contractible subspaces, the product $x_1 \cdots x_n \in \widetilde{K}^0(X)$ is 0 for any $x_1, \ldots, x_n$. Conclude that the product in $\widetilde{K}^0(\Sigma Y)$ is trivial for all $Y$.
    (d) Which properties of $\widetilde{K}^0$ did you use? What can you generalize this proof to?

(4) Show that $\mathbf{C}P^2$ is not homotopy equivalent to $S^2 \vee S^4$ by showing that the Adams operations on their $K$-theories are not the same.