Further reading: [Tho54] again.

The goal for this section is to prove the following theorem and relate it to characteristic numbers of manifolds.

Theorem 10.1. *When $k > n$,*

$$\mathfrak{N}_n \cong \pi_{n+k}\mathrm{Th}(\gamma_k).$$

In Lemma 9.4 we proved that the right-hand side of this equation is well-defined, so this theorem is reasonable as stated.

Remark 10.2. In an abuse of notation, for this section we write G_k for $G_k(\mathbf{R}^N)$ for a large enough N, and write γ_k for the universal bundle γ_{kN}. It will be clear from the discussion that for N large enough the maps will be well-defined and independent of N. This is done so that G_k will be a compact manifold.

We will need a notion of "transverse intersection" from which we will be able to conclude that preimages of manifolds are manifolds.

Definition 10.3. Let $f\colon X \longrightarrow M$ be a C^n map from an n-manifold to a p-manifold. Let $N \subseteq M$ be a submanifold of codimension q. For any point $y \in N$, let T_yM be the tangent space to M at y, and let T_yN be the subspace of T_yM which is the tangent space to N at y. Let $x \in f^{-1}(y)$. We then have a map $df_x\colon T_xX \longrightarrow T_yM \twoheadrightarrow T_yM/T_yN$. We say that f is *transverse to N at y* if this induced map is an epimorphism.

In general, f is transverse to N if it is transverse to N at every point of N. (Note that if $f^{-1}(y) = \emptyset$ then transversality holds automatically.)

Definition 10.4. A homotopy $X \times I \longrightarrow Y$ is an *isotopy* if for each $t \in [0, 1]$ the restriction $X \times \{t\} \longrightarrow Y$ is smooth.

It's important to note that since transversality is a local condition, all that is necessary for the definition is for the spaces to be smooth near the points of interest. In particular, note that for any bundle $E \longrightarrow M$ on a manifold, the space $\mathrm{Th}(E)$ is a manifold away from the basepoint. Thus all consequences of transversality apply to $\mathrm{Th}(E)$ as well.

Transversality satisfies the following properties, proved in [Tho54, Chapter I], which we use without proof:

Theorem 10.5 ([Tho54, Chapter 1])**.** *Let M be a p-manifold and N a paracompact submanifold of codimension q. Let T be a tubular neighborhood of N in M. We assume that X is a smooth n-manifold.*

1. *Let $f\colon X \longrightarrow M$ be any C^n map transverse to N. Then $f^{-1}(N)$ is a C^n $n-q$-submanifold of X.*
2. *Let $f\colon X \longrightarrow M$ be any C^n map. There exists a homeomorphism A of T, arbitrary close to the identity and equal to the identity on ∂T, such that $A \circ f$ is transverse to N. In particular, $(A \circ f)^{-1}(N)$ is a smoothly embedded $n-q$-submanifold of X of class C^n.*
3. *If f is transverse to N, with N compact. Then for any homeomorphism A of T (identity on ∂T) which is sufficiently close to the identity, the map $A \circ f$ is transverse to N and the submanifolds $f^{-1}(N)$ and $(A \circ f)^{-1}(N)$ are isotopic in V.*

In particular, part (3) implies that $f^{-1}(N)$ and $(A \circ f)^{-1}(N)$ are isomorphic.

Similar results also hold when X is a manifold with boundary.

We can now proceed to relate homotopy and cobordism. Just as with the case of classification of vector bundles earlier, we're going to show that we can classify cobordism classes using homotopy classes of maps.

Theorem 10.6. *Let $f, g\colon X \longrightarrow M$ be two C^m maps, where $m \geq n$, and suppose that both are transverse to N. Let $W = f^{-1}(N)$ and $W' = g^{-1}(N)$. If f and g are homotopic then W and W' are cobordant.*

Proof. It is a theorem [Tho54, Lemma IV.5] that if two C^m maps are homotopic then we can assume that the homotopy is also C^m. Thus let $F\colon X \times I \longrightarrow M$ be this homotopy. By the analog of Theorem 10.5 for manifolds with boundary, there exists an A such that $A \circ F$ is transverse to N. Then by Theorem 10.5(3) $V = (A \circ F)|_{X\times\{0\}}^{-1}(N)$ is isotopic to $f^{-1}(N)$ and $V' = (A \circ F)|_{X\times\{1\}}^{-1}(N)$ is isotopic to $g^{-1}(N)$, so it suffices to check that V and V' are cobordant. But $(A \circ F)^{-1}(N)$ is an $n - q + 1$-manifold with boundary $V \amalg V'$, so it is exactly the cobordism we are looking for. $\square$

Thus we see that we have a map

$$[X, M] \longrightarrow \{\text{submanifolds of } X\}/\{\text{cobordism with embedding into } X\}$$

which is well-defined. This motivates the following definition:

Definition 10.7. Let W_0, W_1 be two n-submanifolds of an $n + k$-manifold X. Then they are *L-equivalent in X* if there exists a $n + 1$-manifold Y with boundary $W_0 \amalg W_1$ with an embedding $Y \longrightarrow X \times I$ with $Y \cap X \times \{0\} = W_0$ and $Y \cap X \times \{1\} = W_1$.

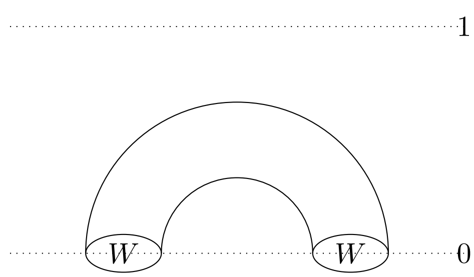
By gluing these embeddings together we see that L-equivalence is an equivalence relation.

We denote by $L_n(X)$ the set of L-equivalence classes of n-submanifolds in X.

Note that L-equivalence is a stronger condition than cobordism. In addition, we note that if W_0 and W_1 are the boundary of a submanifold Z in X then they are L-equivalent: we can use Urysohn's Lemma to construct a function $\varphi\colon Z \longrightarrow [0, 1]$ which is 0 on W_0 and 1 on W_1 and then take the embedding $f\colon Z \longrightarrow X \times I$ given by $f(z) = (i(z), \varphi(z))$.

Lemma 10.8. *For $k > n + 2$, $L_k(S^{n+k})$ is an abelian group. The map φ taking the L-equivalence class of a submanifold to its cobordism class is an isomorphism.*

Proof. When $k > n + 2$ we can always isotope two n-submanifolds of S^{n+k} to be disjoint. Then disjoint union gives a well-defined operation on $L_n(S^{n+k})$. As with cobordisms, the inverse of a submanifold W is itself (via a "horseshoe" cobordism)



with the identity the empty submanifold. Thus $L_n(S^{n+k})$ is an abelian group. Since L-equivalence implies cobordism, φ is well-defined; with this operation it is clearly a homomorphism.

Since any manifold of dimension n can be embedded into S^{n+k} (as $n + k > 2n$) φ is surjective. Now suppose that a submanifold W of S^{n+k} is null-cobordant; thus there exists a manifold B of dimension $n + 1$ with W as the boundary. Since $n + k > 2(n + 1)$, there exists an embedding $f\colon B \longrightarrow S^{n+k}$. By Urysohn's lemma there exists a map $\psi\colon B \longrightarrow [0, 1]$ such that $\psi^{-1}(0) = W$. Then the embedding $B \longrightarrow S^{n+k} \times [0, 1]$ given by $x \longmapsto (f(x), \psi(x)/2)$ shows that W is L-equivalent to the empty set, as desired. Thus φ is injective, and thus an isomorphism. $\square$

For an $n + k$-manifold X we define a map

$$J\colon L_n(X) \longrightarrow [X, \mathrm{Th}(\gamma_k)].$$

Choose an embedding of X into $\mathbf{R}^N$, for some large N. Let W be a k-submanifold of X. For any $w \in W$, consider the subspace $(T_w W)^\perp \cap T_w X$; this is a k-plane, which we can consider to be a point of G_k. This gives us a map $f\colon W \longrightarrow G_k$. Let N be a tubular neighborhood of W; we can think of this as the disk bundle of the bundle $f^*(\gamma_k)$. Then f induces a map $\widetilde{f}\colon \mathrm{Th}(f^*(\gamma_k)) \longrightarrow \mathrm{Th}(\gamma_k)$. We can define a map $f'\colon X \longrightarrow \mathrm{Th}(\gamma_k)$ by

$$f'(x) = \begin{cases} * & \text{if } x \notin N \\ \widetilde{f}(x) & \text{if } x \in N. \end{cases}$$

Since the boundary of N maps to the distinguished point in $\mathrm{Th}(\gamma_k)$ this is continuous.

Lemma 10.9. *The map J is independent of the choice of embedding of X into $\mathbf{R}^N$.*

Proof. Suppose that $i_0, i_1\colon X \longrightarrow \mathbf{R}^N$ are two different embeddings. For N large enough we can assume that their images do not intersect, and that there is an embedding of $X \times I \longrightarrow \mathbf{R}^N$ such that the restriction to $X \times \{0\}$ is i_0 and the restriction to $X \times \{1\}$ is i_1 (possibly up to some translation).

Let W be some k-submanifold of X. The embedding above restricts to an embedding of $W \times I$ in $\mathbf{R}^N$. Let N be a tubular neighborhood of $W \times I$ in this embedding. Assuming that X is embedded orthogonally to its boundary, this is going to be orthogonal to the boundary; thus $N \cap X \times \{0\}$ will be a tubular neighborhood of $i_0(W)$ and $N \cap X \times \{1\}$ will be a tubular neighborhood of $i_1(W)$. We can apply the construction above to this N and embedding of $W \times I$ to produce a map $X \times I \longrightarrow \mathrm{Th}(\gamma_k)$ which will restrict to the map constructed for W under i_0 and i_1, respectively. Thus the two produced maps are homotopic, as desired. □

Similarly we can show that the embedding is independent of choice of tubular neighborhood.

By a similar proof, we can see that L-equivalent manifolds give homotopic maps (since we never actually used that $W \times I$ is a product, but rather just that it is an embedded manifold with the right boundary), so J is well-defined.

We now claim that J is a bijection. We have a map

$$G\colon [X, \mathrm{Th}(\gamma_k)] \longrightarrow L_{n-k}(X)$$

defined by $f \longmapsto f^{-1}(G_k)$, where G_k is considered to be the zero section of γ_k and f is transverse to G_k. Note that if we assume that G is well-defined then by the construction of f, $G \circ J$ is the identity.

Again assuming that G is well-defined, note that the only maps that map to the class of the empty submanifold are the null-homotopic ones. Indeed, suppose that $f\colon X \longrightarrow \mathrm{Th}(\gamma_k)$ has $f^{-1}(G_k) = \emptyset$. Then for all x, $f(x) = (\omega, v)$ with $v \in \omega$ and $0 < \|v\| < 1$ or $f(x) = *$. We define a homotopy $F\colon X \times I \longrightarrow \mathrm{Th}(\gamma_k)$ by

$$F(x,t) = \begin{cases} * & \text{if } f(x) = * \text{ or } t = 1 \\ (\omega, ((1-t) + \frac{t}{\|v\|})v) & \text{if } f(x) = (\omega, v). \end{cases}$$

Thus f is null-homotopic.

Lemma 10.10. *Let $f, f'\colon X \longrightarrow \mathrm{Th}(\gamma_k)$ be two maps which are homotopic and transverse to G_k. Then $f^{-1}(G_k)$ is L-equivalent to ${f'}^{-1}(G_k)$.*

Proof. Let $F\colon X \times I \longrightarrow \mathrm{Th}(\gamma_k)$ be the homotopy between f and f'. By [Tho54, Lemma IV.5], we can assume that this homotopy is smooth. By Theorem 10.5(2) we can also assume that the homotopy is transverse to G_k. Then $F^{-1}(G_k)$ is a submanifold whose boundary lies in $X \times \{0,1\}$, with the intersection with $X \times \{0\}$ being $f^{-1}(G_k)$ and the intersection with $X \times \{1\}$ being ${f'}^{-1}(G_k)$. Thus $f^{-1}(G_k)$ and ${f'}^{-1}(G_k)$ are L-equivalent, as desired. □

Thus G is well-defined. It may still be the case that J is not a bijection, however. We will show that it is via a somewhat roundabout method.

Remark 10.11. We will only prove that J is a bijection when X is a sphere. In fact, J is always a bijection, as it can be shown directly (by constructing appropriate homotopies) that J is surjective. Moreover, for $k > n+2$ the set $[X, \mathrm{Th}(\gamma_k)]$ will always be a group, so the following proof will work for any X in those dimensions. However, the first approach fails to discuss the group structure we are interested in, and the second approach requires too many preliminaries. For a complete proof of both perspectives, see [Tho54, Chapter IV].

First, note that for any X, $L_n(X)$ is a group with the operation being disjoint union and identity the class of the empty manifold. Now suppose that $X = S^{n+k}$. Then we have a group operation on $[S^{n+k}, \mathrm{Th}(\gamma_k)]$ given by taking maps $f, g\colon S^{n+k} \longrightarrow \mathrm{Th}(\gamma_k)$ to the composition

$$S^{n+k} \longrightarrow S^{n+k} \vee S^{n+k} \xrightarrow{f \vee g} \mathrm{Th}(\gamma_k) \vee \mathrm{Th}(\gamma_k) \xrightarrow{\nabla} \mathrm{Th}(\gamma_k).$$

This is just the usual operation on homotopy groups, so it gives $[S^{n+k}, \mathrm{Th}(\gamma_k)]$ a group structure. Note that

$$(f+g)^{-1}(G_k) = f^{-1}(G_k) \amalg g^{-1}(G_k).$$

Thus G is a group homomorphism. Conversely, if we take the union of disjoint n-submanifolds of S^{n+k} then the tubular neighborhood of the union is the union of two tubular neighborhoods. Tracing through the construction of J shows that J is a homomorphism as well. (Note that for this to always be possible we want $2k > n$.)

Since J is injective and $G \circ J$ is the identity, G is surjective; thus G (and therefore J) will be an isomorphism if it has trivial kernel. We showed that this is the case above.

We have thus shown that

$$L_n(S^{n+k}) \cong \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

By Lemma 10.8, $L_n(S^{n+k}) \cong \mathfrak{N}_n$. Putting these together we get

$$\mathfrak{N}_n \cong L_n(S^{n+k}) \cong \pi_{n+k}\mathrm{Th}(\gamma_k) \qquad \text{when } k > n,$$

which is eactly the statement of Theorem 10.1.

We would like to relate this to classifying cobordism classes of manifolds. Suppose that we have an n-manifold M; we may choose an embedding into S^{n+k}, for k sufficiently large. By the same algorithm as above, we thus get a homotopy class of maps $f\colon S^{n+k} \longrightarrow \mathrm{Th}(\gamma_k)$. By the Theorem 10.1, we know that M is the boundary of a manifold if and only if this map is null-homotopic. We would like to show that this map is null-homotopic exactly when all of the characteristic numbers of M are zero.

We have that $M = f^{-1}(G_k)$. Thus f restricts to a map $\widetilde{f}\colon M \longrightarrow G_k$. By our classification of bundles, we know that this is the classifying map of some bundle. From the construction of the map $M \longrightarrow G_k$, we can see that this is actually the classifying map of the normal bundle to the embedding of M into S^{n+k}. If there was some element $\alpha \in H^n(G_k)$ such that $\widetilde{f}^*\alpha \neq 0 \in H^n(M)$, then $\widetilde{f}^*\alpha \frown [M] \neq 0$. If we assume that α is a monomial in the generators of $H^n(G_k)$ it follows that there exists a characteristic number of the normal bundle to M (and thus a characteristic number of M) which is nonzero. Thus it suffices to check that f is null-homotopic exactly when all elements $\alpha \in H^n(G_k)$ pull back to 0 along f.

Clearly, the forwards direction is true: if f is null-homotopic then we can just compute the pullbacks of elements along a constant map, and they all map to 0 trivially. Thus all that remains is the backwards direction. We have the following commutative diagram:

$$\begin{tikzcd}
N \arrow[r, hook] & S^{n+k} \arrow[r, "f"] & \mathrm{Th}(\gamma_k) \\
M \arrow[u, hook] \arrow[rr, "\widetilde{f}"] & & G_k \arrow[u, hook]
\end{tikzcd}$$

The vertical maps are the inclusions of the maps as zero sections of the bundles. They are the maps that induce the Thom isomorphism, which is natural; thus we get a diagram in cohomology

$$\begin{tikzcd}
\widetilde{H}^{n+k}(N) & \underbrace{\widetilde{H}^{n+k}(S^{n+k})}_{\mathbb{Z}/2} \arrow[l] & \widetilde{H}^{n+k}(\mathrm{Th}(\gamma_k)) \arrow[l, "f^*"'] \\
\widetilde{H}^{n}(M_+) \arrow[u, "\cong"] & & \widetilde{H}^{n}(G_{k+}) \arrow[ll] \arrow[u, "\cong"']
\end{tikzcd}$$

where the vertical maps are Thom isomorphisms. The map across the bottom is zero exactly when the map across the top is zero. It is easy to check that the left-hand map is nonzero; thus all we need to check is that f^* is zero exactly when f is null-homotopic.

Lemma 10.12. *The map f^* is zero only if f is null-homotopic.*

We omit the proof of this: the proof is somewhat technical and would take us even further afield that we have already gone.