12. Topological K-theory

Further reading: [Hat, Chapter 2], [Ati89]

One interesting side effect of this calculation is that now we can construct another cohomology theory. Recall that we proved that we can get a cohomology theory from a sequence of spaces $X_0, X_1, \ldots$ together with equivalences $X_i \xrightarrow{\sim} \Omega X_{i+1}$ for all i. One example that we knew came from Eilenberg–MacLane spaces, but we did not have another good example. Now we can construct one using Bott periodicity.

For traditions sake, we will work with the form of Bott periodicity that states that $\mathbb{Z} \times BU \simeq \Omega U \simeq \Omega^2(\mathbb{Z} \times BU)$. We can thus conclude that

$$K := \mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots$$

is a spectrum. Here the first map is given by the Bott map, the second by the identity, the third by the Bott map, and so on. We may therefore ask: which cohomology theory is represented by this spectrum?

Theorem 12.1. *For any compact connected space X,*

$$\widetilde{K}^0(X_+) = \{\textit{free ab. gp. generated by iso classes of vector bundles on } X\}/[E \oplus E'] = [E] + [E'].$$

For general i, $K^i(X_+) = K^0(\Sigma^i X_+)$.

Proof. We prove the second part first. Note that for all $i \geq 0$ we can define the $-i$-th space of the spectrum by defining K_{-i} to be $\mathbb{Z} \times BU$ if i is even an U if i is odd. Then we have weak equivalences $K_i \longrightarrow \Omega K_{i+1}$ for all i. Then we have

$$\widetilde{K}^i(X_+) = [X_+, K_i] \cong [X_+, K_{-i}] \cong [X_+, \Omega^i(\mathbb{Z} \times BU)] \cong [\Sigma^i X_+, \mathbb{Z} \times BU].$$

Thus if we can prove the first part of the theorem, the second part follows.

Let G be the group given by the right-hand side of the expression in the theorem. We define a homomorphism $\varphi\colon \widetilde{K}^0(X_+) \longrightarrow G$ by the following. Consider a class $\alpha \in [X_+, \mathbb{Z} \times BU]$. Since X is connected, the image of the X-compenent lies in a single component $\{i\} \times BU$. Since X is compact and $BU = \operatorname{colim}_n BU(n)$, we know that α is represented by an $f\colon X \longrightarrow \{i\} \times BU$ which factors through some finite stage $BU(n)$. We then define

$$\varphi[f, i] = [f^*\gamma_n] - [\epsilon^{n-i}].$$

We define $[\epsilon^{-j}] = -[\epsilon^j]$.

We must check that φ is well-defined. Note that changing $f\colon X \longrightarrow BU(n)$ up to homotopy changes $f^*\gamma_n$ by an isomorphism, so that does not affect φ. However, there is a further wrinkle. In the discussion above, we chose n such that f factors through $BU(n)$. However, n is not well-defined; if f factors through $BU(n)$ it also factors through $BU(m)$ for $m \geq n$. However, the map $BU(n) \longrightarrow BU(n+1)$ is given by the map $U(n) \longrightarrow U(n+1)$ adding a 1 in the lower-right corner of the matrix. The pullback of a vector bundle along this map takes γ_{n+1} to $\gamma_n \oplus \epsilon^1$, as we saw in the proof of Lemma 5.4. Thus if we replace n by $m \geq n$ we replace $f^*\gamma_n$ by $f^*\gamma_n \oplus \epsilon^{m-n}$. But then, inside G we have

$$\begin{aligned}[f^*\gamma_n \oplus \epsilon^{m-n}] - [\epsilon^{m-i}] &= [f^*\gamma_n] \oplus [\epsilon^{m-n}] - [\epsilon^{n-i}] - [\epsilon^{m-n}] \\ &= [f^*\gamma_n] - [\epsilon^{n-i}].\end{aligned}$$

Thus $\varphi[f, i]$ is well-defined.

The addition is given by the H-space structure on $\mathbb{Z} \times BU$. As discussed in the previous section, this H-space structure is given by the sum on the $\mathbb{Z}$-components and the Whitney sum of representing bundles on the BU-component. Thus we see that two elements of $[X, \mathbb{Z} \times BU]$ represented by $[f,i]$ and $[f',j]$ have sum represented by $[f\oplus f', i+j]$. Thus we have

$$\begin{aligned}\varphi[f\oplus f', i+j] &= [(f\oplus f')^*\gamma_{n+m}] - [\epsilon^{n+m-i-j}] = [f^*\gamma_n \oplus f'^*\gamma_m] - [\epsilon^{n-i}\oplus\epsilon^{m-j}] \\ &= \varphi[f,i] + \varphi[f',j].\end{aligned}$$

We now need to check that φ is an isomorphism. First, we check that φ is surjective. Let

$$\alpha = \sum_{i=1}^{n} a_i[E_i]$$

be any element of G. By using the relation in G, we can rewrite this element as $[E]-[E']$, where E is the sum of all E_i with positive coefficient (taken sufficiently many times) and E' is the sum of all E_i with negative coefficient. Since X is compact, there exists an E'' such that $E'\oplus E'' \cong \epsilon^m$ for some m. Then

$$[E]-[E'] = [E\oplus E''] - [E'\oplus E''] = [E\oplus E''] - [\epsilon^m].$$

Let $n = \dim E\oplus E''$, let $f\colon X \longrightarrow BU(n)$ be the classifying map of $E\oplus E''$, and let $i = n-m$. Then the map

$$X \xrightarrow{f} \{i\}\times BU(n) \longrightarrow \{i\}\times BU$$

is mapped to α, as desired.

Now we consider injectivity. Let $f\colon X\longrightarrow \{i\}\times BU$ be a map such that $\varphi[f,i]=0$; assume that f factors through $\{i\}\times BU(n)$. Since $\varphi[f,i]=0$, there exists a bundle E such that $f^*\gamma_n\oplus E \cong \epsilon^{n-i}\oplus E$. Note that the dimension of the left-hand side is $n+\dim E$ and the dimension of the right-hand side is $n-i+\dim E$, so we must have $i=0$. In addition, since X is compact there exists a bundle E' such that $E\oplus E'\cong\epsilon^m$; thus we have

$$f^*\gamma_n\oplus\epsilon^m \cong \epsilon^{n+m}.$$

(When a bundle satisfies such an equation we say that it is *stably trivial*.) However, this means that the map

$$X \xrightarrow{f} BU(n) \longrightarrow BU(n+m)$$

classifies a trivial bundle, and thus is null-homotopic. But this implies that $X \xrightarrow{f} BU(n) \longrightarrow BU$ is null-homotopic, and thus $[f]$ was trivial, as desired. Thus φ is injective. □

Remark 12.2. As usual, $\widetilde{K}^*(X_+)$ is written $K^*(X)$.

We enumerate some properties of $K(X)$ which follow directly from Theorem 12.1.

(1) When X is already pointed, $\widetilde{K}(X)$ is subgroup of $\widetilde{K}(X_+)$ consisting of those maps which always hit the $\{0\}\times BU$ component. Then using the theorem we can write elements of $\widetilde{K}(X)$ as formal differences $[E]-[\epsilon^{\dim E}]$. This is exactly the kernel of the map $K^0(X_+)\longrightarrow K^0(*_+)$ induced by the inclusion $*\longrightarrow X$, where $*$ is the basepoint of X.

(2) Two vector bundles E and E' are equal in $\widetilde{K}^0(X)$ exactly when there exist trivial bundles ϵ^k and $\epsilon^{k'}$ such that $E\oplus\epsilon^k \cong E'\oplus\epsilon^{k'}$. In particular, all trivial bundles are 0 inside $\widetilde{K}^0(X)$.

(3) Moreover, let $[E]\in\widetilde{K}^0(X)$. Let E' be a bundle such that $E\oplus E'\cong\epsilon^n$. Then

$$[E]+[E'] = [E\oplus E'] = [\epsilon^n] = 0,$$

so $[E] = -[E']$.

(4) Since $\widetilde{K}$ is a cohomology theory, we know that for any cofiber sequence $A\hookrightarrow X\longrightarrow X/A$ we have a long exact sequence

$$\cdots \longrightarrow \widetilde{K}^i(X/A) \longrightarrow \widetilde{K}^i(X) \longrightarrow \widetilde{K}^i(A) \longrightarrow \widetilde{K}^{i+1}(X/A) \longrightarrow \cdots.$$

(5) Suppose that X is pointed. For all i,

$$\widetilde{K}^i(X) \stackrel{\text{def}}{=} [X, K_i] \cong [X, \Omega^2 K_i] \cong [\Sigma^2 X, K_i] = \widetilde{K}^{i-2}(X).$$

Thus $\widetilde{K}^*(X)$ is 2-periodic for all X. This statement can be proven directly, and is often referred to as "Bott periodicity."

Consider $K^0(X)$. The addition comes from the addition of vector bundles, but there is also a natural *multiplication* of vector bundles: the tensor product. This structure extends to $K^0(X)$:

$$([E]-[\epsilon^{\dim E-k}])([E']-[\epsilon^{\dim E-k'}]) = [E\otimes E]-[\epsilon^{\dim E-k}\otimes E']-[E\otimes\epsilon^{\dim E'-k'}]+[\epsilon^{(\dim E-k)(\dim E'-k')}].$$

Note that when $k = k' = 0$ then the total dimension of this is

$$\dim E \dim E' - \dim E \dim E' - \dim E \dim E' + \dim E \dim E' = 0,$$

so this multiplication is well-defined on $\widetilde{K}^0$, as well.

Example 12.3. Let's compute the ring structure on $K^0(S^2) \cong \widetilde{K}^0(S^2_+)$. First, observe that

$$K^0(S^2) \cong [S^2_+, \mathbb{Z} \times BU] \cong \mathbb{Z} \oplus [S^2, BU] \cong \mathbb{Z} \times \mathbb{Z}.$$

Thus the additive structure is $\mathbb{Z} \times \mathbb{Z}$. The first $\mathbb{Z}$ is generated by the trivial bundles; the second $\mathbb{Z}$ is generated by the canonical line bundle of $\mathbf{C}P^1$, which we call H.** We can check that these are different using characteristic classes, since all of the characteristic classes of trivial bundles are 0, while H has a nontrivial first Chern class.

We claim that H satisfies the relation

$$(H \otimes H) \oplus 1 \cong H \oplus H.$$

Consider a map $S^2 \longrightarrow BU(2)$; since $S^2 \cong \Sigma S^1$, this is uniquely determined by a map $S^1 \longrightarrow \Omega BU(2) \simeq U(2)$. Thus a vector bundle on S^2 is uniquely determined by the homotopy class of a map $S^1 \longrightarrow U(2)$; this is called the *clutching function.*†† To get this function explicitly, we think of S^2 as $D^2 \cup_{S^1} D^2$. Since D^2 is contractible, any vector bundle is isomorphic to a trivial bundle on D^2. Thus we can take the two D^2's as being our atlas of trivializations. We then have a transition function, which for every point in S^1 gives a linear map (which can be taken to be in $U(2)$). This map is the clutching function.

Note that the clutching function for a bundle which is the Whitney sum of two bundles is the block diagonal of the clutching functions for the two bundles. Similarly, the clutching function for the tensor product of two bundles is the tensor product of the two clutching functions. Thus to compute the clutching functions of the bundles we are interested in, it suffices to compute the clutching function of H.

Consider a point in $\mathbf{C}P^1$ as a pair $[z_0 \colon z_1]$. The line above this point is the line consisting of the points $(\lambda z_0, \lambda z_1)$. Thus inside the unit disk we can assume that $z_1 = 1$ and write the trivialization of the bundle as $[z_0/z_1 \colon 1] \times \mathcal{C} \longrightarrow (\lambda z_0/z_1, \lambda)$. Outside the unit disk we can assume that $z_0 = 1$ and write the trivialization of the bundle as $[1 \colon z_1/z_0] \times \mathcal{C} \longrightarrow (\lambda, \lambda z_1/z_0)$. When $|z_1/z_0| = 1$, the transition function takes $([z_0/z_1 \colon 1], \lambda)$ to $([1 \colon z_1/z_0], \lambda)$, so this function takes $z \in S^1$ to the function $(z) \in U(1)$.

Using this, we see that the clutching functions for $(H \otimes H) \oplus 1$ and $H \oplus H$ are

$$\begin{pmatrix} z^2 & \\ & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} z & \\ & z \end{pmatrix}.$$

**We could also call it γ_{11}, but this could cause confusion with the 11-dimensional canonical bundle. We also prefer H as it is consistent with both [Ati89] and [Hat].

††Note that nothing stated here depends on the fact that we're working with S^2; this approach actually works for all S^k.

To show that these are isomorphic we just need to show that these clutching functions are homotopic. However, the first one is the pointwise multiplication in ΩU of (z) with itself. The second one is loop addition. As discussed before, these are homotopic.

From this discussion we have that $H^2 + 1 = 2H$, or in other words that $(H-1)^2 = 0$. We thus have a natural homomorphism $\mathbb{Z}[H]/(H-1)^2 \longrightarrow K(S^2)$. Since this homomorphism is onto and since the rank of the source is equal to the rank of the target, it must be an isomorphism.

Note that if we think of $K(S^2) \cong \mathbb{Z}\times\mathbb{Z}$ as generated by $[H]-[\epsilon^1]$ and $[\epsilon^1]$ then $\widetilde{K}(S^2)$ is generated by $H-1$, with the relation $(H-1)^2 = 0$.

Remark 12.4. By tracing through this and the definition of the Bott map we can show explicitly that the map

$$\widetilde{K}^0(X) \longrightarrow \widetilde{K}^0(\Sigma^2 X)$$

can be defined to be multiplication by $H-1$. Thus we can extend the ring structure on $\widetilde{K}^0(X)$ to $\widetilde{K}^*(X)$.