15. The J-homomorphism

Further reading: [HatB, Chapter 4]

Bott periodicity tells us that homotopy groups of O are 8-periodic. Using the fact that

$$\pi_i X = [\Sigma^i S^0, X] = [S^0, \Omega^i X] = \pi_0 \Omega^i X,$$

we can use Bott periodicity to compute that that the groups $\pi_i O$ are

$$\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \dots.$$

If we restrict to SO this replaces the first of these groups with 0 but maintains the periodicity. If we could find a map $SO \longrightarrow \mathbf{S}$ where we could control the images of these groups, it would give us nice elements in the homotopy groups of $\mathbf{S}$. We will construct such a map now; the induced homomorphism on homotopy groups is called the *J-homomorphism*.

Definition 15.1. The *join* of two spaces X and Y, written $X * Y$ is the space $X \times Y \times I/\sim$, where $(x, y_0, 0) \sim (x, y_1, 0)$ and $(x_0, y, 1) \sim (x_1, y, 1)$ for all $x, x_0, x_1 \in X$ and $y, y_0.y_1 \in Y$.

Note that $CX \cong \{*\} * X$ and that $SX \cong S^0 * X$. More generally, $S^i \cong \underbrace{S^0 * S^0 * \cdots * S^0}_{i+1}$ and thus $S^a * S^b \cong S^{a+b+1}$.

Definition 15.2. Suppose that we are given a map $f\colon X \times Y \longrightarrow Z$. Note that $S(X \times Y)$ is the space $X \times Y \times I/\sim$, where $(x, y, 0) \sim (x', y', 0)$ and $(x, y, 1) \sim (x', y', 1)$ for all $x, x' \in X$ and $y, y' \in Y$. Note that from this definition, the map

$$X \times Y \times I \longrightarrow S(X \times Y)$$

factors through $X * Y$. Write $h_{X,Y}\colon X * Y \longrightarrow S(X \times Y)$.

The *Hopf construction* of f is the map $\widetilde{f}\colon X * Y \longrightarrow SZ$ given by

$$X * Y \longrightarrow S(X \times Y) \longrightarrow SZ.$$

Consider the group $SO(n)$. An element $\gamma \in SO(n)$ acts on $\mathbf{R}^n$ and preserves norms, and thus gives a map $\widetilde{\gamma}\colon S^{n-1} \longrightarrow S^{n-1}$. Thus an element $[f] \in \pi_i(SO(n))$ is a map $S^i \times S^{n-1} \longrightarrow S^{n-1}$ which takes the basepoint of S^i to the identity. Then the Hopf construction of f gives us a map

$$Jf\colon S^{i+n} \cong S^i * S^{n-1} \xrightarrow{\widetilde{f}} S(S^{n-1}) \cong S^n.$$

If we change f by a homotopy then this map changes by a homotopy, so it gives a class $[Jf] \in \pi_{n+i} S^n$. If we include $SO(n)$ into $SO(n+1)$ and see how the image changes we note that $\widetilde{\gamma}$ becomes $S\widetilde{\gamma}$, and by the definition of the Hopf construction $[Jf]$ maps to $[SJf]$. Thus we actually get a well-defined element in $J[f] \in \pi_i^S S^0$.

Lemma 15.3. *J is a homomorphism.*

Proof. It suffices to check that J commutes with addition for any individual value of n. The addition of two elements $[f], [g] \in \pi_i SO(n)$ gives a composition

$$S^i \longrightarrow S^i \vee S^i \xrightarrow{f \vee g} SO(n) \vee SO(n) \xrightarrow{\nabla} SO(n).$$

Note that $\vee$ and $*$ commute past one another, so we obtain a map

$$S^i * S^{n-1} \longrightarrow (S^i * S^{n-1}) \vee (S^i * S^{n-1}) \xrightarrow{\widetilde{f} \vee \widetilde{g}} S^{n-1} \vee S^{n-1} \xrightarrow{\nabla} S^{n-1}.$$

This is exactly the map $Jf + Jg$. ∎

The J-homomorphism is the first example of obtaining an "approximation" to the sphere spectrum using a periodic spectrum. The image of the J-homomorphism is the "first chromatic layer" of the sphere spectrum.

Note that we cna immediately see that when i is even this has trivial image. When $i = 1 \pmod 8$ this is a map $\mathbb{Z}/2 \longrightarrow \pi_i^s S^0$, so it has image at most a $\mathbb{Z}/2$. The most interesting versions of this homomorphisms are when $i \equiv 3 \pmod 4$, when this is a map $\mathbb{Z} \longrightarrow \pi_i^s S^0$; this map will have finite image (since stable homotopy groups of spheres are finite above level 0), but if we can identify the size of the image it will give us a lower bound on the size of the homotopy group.

In fact, we have an even stronger result:

Theorem 15.4 (Adams)**.** *The group $\pi_n^s(S^0)$ is the direct sum of the image of $J|_{SO}\colon \pi_n(SO(k)) \longrightarrow \pi_{n+k}(S^n)$ (for k large enough) and the kernel of the Adams e-invariant.*

This is proved in a series of papers by Adams on the J-homomorphism: [Ada63, Ada65A, Ada65B, Ada66].

Thus our next step is to try to figure out how to find some bounds on the image of the J-homomorphism.

To do this, we will need the *Chern character*. The abomination is convenient because it takes Whitney sums of bundles on X to products of classes; however, it does not take products to anything simple. The Chern character is designed so that it is actually a homomorphism of rings $K^*(X) \longrightarrow H^*(X;\mathbb{Q})$. Its disadvantage is that it doesn't work for integer coefficients; it is necessary to pass to rational coefficients to make it reasonable.

To begin, let us first explore line bundles in a bit more depth.

Proposition 15.5. *The set* $\mathrm{Vect}^1(X)$ *is a group under tensor product. With this structure, $c_1\colon \mathrm{Vect}(X) \longrightarrow H^2(X;\mathbb{Z})$ is a homomorphism, and is an isomorphism if X is a CW complex.*

Proof. To check that $\mathrm{Vect}^1(X)$ is a group, it suffices to check that inverses exist. For this, suppose that we are given a vector bundle E. Let $\{U_\alpha\}$ be an open cover of X such that E is trivial over each U_α, and let $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow GL_1(\mathbf{C})$ be the transition functions. Let E' be the vector bundle given by trivial bundles over U_α and gluing functions $g'_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow GL_1(\mathbf{C})$ given by $g'_{\alpha\beta}(u) = g_{\alpha\beta}(u)^{-1}$. In such a presentation, the tensor product of two line bundles has transition functions which are products of the transition functions of the components (since elements in $GL_1(\mathcal{C})$ commute past each other). Thus $E' \otimes E \cong \epsilon^1$, as desired.

Note that $BU(1) \cong \mathbf{C}P^\infty$ by construction. Thus $\mathrm{Vect}^1(X) \cong [X, \mathbf{C}P^\infty]$. In addition, we can show that $\mathbf{C}P^\infty \cong K(\mathbb{Z},2)$. Thus $[X, \mathbf{C}P^\infty] \cong [X, K(\mathbb{Z},2)] \cong H^2(X)$. The map c_1 is given by pullback from the cohomology of H^2, as is the map $\mathrm{Vect}^1(X) \longrightarrow H^2(X)$, so this is a bijection of sets.

Thus it remains to show that c_1 is a homomorphism, or in other words that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. Let L be the canonical line bundle. Note that it is in fact sufficient to prove that $c_1(L \otimes L) \cong c_1(L) + c_1(L) \in H^2(BU(1))$, since the general fact will follow from naturality. □

Now we know that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. We want to define the Chern character to take $L_1 \otimes L_2$ to a prouct, so we define

$$ch(L) = e^{c_1(L)}.$$

Then on line bundles products are taken to products. We now want to extend this definition to all bundles. Write $t_i = c_1(L_i)$ for conciseness. Note that we want

$$ch(L_1 \oplus \cdots \oplus L_n) = \sum_{i=1}^{n} ch(L_i) = \sum_{i=1}^{n} e^{t_i} = \dim E + \sum_{j=1}^{\infty} \frac{1}{j!}(t_1^j + \cdots + t_n^j)$$
$$= \dim E + \sum_{j=1}^{\infty} \frac{1}{j1} s_j\big(\sigma_1(t_1, \dots, t_n), \dots, \sigma_j(t_1, \dots, t_n)\big).$$

On the other hand, from the definition of the Chern class of a Whitney sum we know that

$$c_j(E) = \sigma_j(t_1, \dots, t_n).$$

Thus we can rewrite the last formula above as

$$\sum_{j=1}^{\infty} s_j(c_1(E), \dots, c_j(E)).$$

This formula is well-defined for all bundles E, so we can define

$$ch(E) \stackrel{\text{def}}{=} \dim E + \sum_{j=1}^{\infty} \frac{1}{j!} s_j(c_1(E), \dots, c_j(E)).$$

Lemma 15.6. *ch is a ring homomorphism $K^0(X) \longrightarrow H^*(X; \mathbb{Q})$.*

Proof. First note that in order to verify that $ch(E \oplus E') = ch(E) + ch(E')$ and $ch(E \otimes E') = ch(E)ch(E')$ it suffices to check it works on sums of line bundles. Indeed, the proof of the Splitting Principle works with any cohomology coefficients, so in particular it works with rational coefficients. Thus we can just check this when both E and E' are sums of line bundles. The sum formula works by construction. The product formula works because

$$ch(E \otimes E') = ch\left(\bigoplus_{j=1}^{m} L_j \otimes \bigoplus_{k=1}^{n} L'_k\right) = ch\left(\bigoplus_{j,k} L_j \otimes L'_k\right)$$
$$= \sum_{j,k} ch(L_j \otimes L'_k) = \sum_{j,k} ch(L_j)ch(L'_k) = ch(E)ch(E').$$

□

By naturality, there is also a reduced version $ch\colon \widetilde{K}^0(X) \longrightarrow \widetilde{H}^*(X; \mathbb{Q})$.

Remark 15.7. The Chern character can be defined as a map of E_∞-ring spectra, thus giving the previous definition by definition. This approach takes more work and theory to define and compute properly, so we avoid it here.

Before we end this section, a couple of useful propositions:

Proposition 15.8 ([HatB, Proposition 4.3])**.** *The map $ch\colon \widetilde{K}^0(S^{2n}) \longrightarrow \widetilde{H}^*(S^{2n}; \mathbb{Q})$ is injective with image $H^*(S^{2n})$.*

Proposition 15.9 ([HatB, Proposition 4.5])**.** *The map $K^*(X) \otimes \mathbb{Q} \longrightarrow H^*(X; \mathbb{Q})$ induced by the Chern character is an isomorphism when X is a finite CW complex.*