2. Grassmannians

Reference for this section: [Hat, Section 1.2], [MS74, Section 5]

We now turn to a vital example of a vector bundle which will play a significant role in our future discussions.

The *Grassmannian* $G_n(\mathbf{R}^k)$ is the manifold of n-planes in $\mathbf{R}^k$. As a set it consists of all n-dimensional subspaces of $\mathbf{R}^k$. To describe it in more detail we must first define the Steifel manifold.

Definition 2.1. The *Stiefel manifold* $V_n(\mathbf{R}^k)$ is the set of orthogonal n-frames of $\mathbf{R}^k$. Thus the points of it are n-tuples of orthonormal vectors in $\mathbf{R}^k$. We can think of $V_n(\mathbf{R}^k)$ as a subset of $(S^{k-1})^n$; we let it inherit its topology from this space. We also note that it is a closed subspace of a compact space, so $V_n(\mathbf{R}^k)$ is compact.

We have a map $V_n(\mathbf{R}^k) \longrightarrow G_n(\mathbf{R}^k)$ which takes each n-frame to the subspace it spans. We let $G_n(\mathbf{R}^k)$ be construted via the quotient topology from $V_n(\mathbf{R}^k)$. Thus $G_n(\mathbf{R}^k)$ is also compact.

Lemma 2.2. *$G_n(\mathbf{R}^k)$ is Hausdorff.*

Proof. To prove that $G_n(\mathbf{R}^k)$ is Hausdorff it suffices to show that for any two n-planes $\omega_1, \omega_2 \in G_n(\mathbf{R}^k)$ there exists a continuous function $f\colon G_n(\mathbf{R}^k) \longrightarrow \mathbf{R}$ such that $f(\omega_1) \neq f(\omega_2)$. For any point $p \in \mathbf{R}^k$, let $f_p\colon G_n(\mathbf{R}^k) \longrightarrow \mathbf{R}$ set $f_p(\omega)$ to be the Euclidean distance from p to the ω. This is continuous because for any n-frame $(v_1, \ldots, v_n) \in V_n(\mathbf{R}^k)$ representing ω we have

$$f_p(\omega) = \sqrt{p \cdot p - (p \cdot v_1)^2 - \cdots - (p \cdot v_n)^2}.$$

This is clearly a continuous function $V_n(\mathbf{R}^k) \longrightarrow \mathbf{R}$ which gives the same value on each preimage of ω, so it is a continuous function $\mathcal{G}_n(\mathbf{R}^k) \longrightarrow \mathbf{R}$. Now let p be any point in ω_1 which is not in ω_2. Then $f_p(\omega_1) = 0$ but $f_p(\omega_2) \neq 0$, and we are done. $\square$

Lemma 2.3. *$G_n(\mathbf{R}^k)$ is a manifold.*

Proof. Let ν be an n-plane in $\mathbf{R}^k$, and let $\nu^\perp$ be its orthogonal complement (of dimension $k-n$). Then the set of n-planes in $\mathbf{R}^k$ which do not intersect $\nu^\perp$ is homeomorphic to the set of graphs of linear maps $\nu \longrightarrow \nu^\perp$, which is the space of $n \times (k-n)$ matrices. This gives a homeomorphism of a neighborhood of ν to $\mathbf{R}^{n(k-n)}$, as desired. $\square$

Since $G_n(\mathbf{R}^k)$ is Hausdorff we can try to construct a CW structure on it. This is relatively simple once we figure out the correct cells to look at. Let $p_i\colon \mathbf{R}^k \longrightarrow \mathbf{R}^i$ be the projection onto the first i coordinates; thus p_k is the identity and p_0 is the trivial map to the point. As i goes from k to 0 the dimension of the image of an n-plane $\omega \in G_n(\mathbf{R}^k)$ drops from n to 0; let σ_i be the smallest integer such that $\dim p_i(\omega) = i$. The sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ is called a *Schubert symbol.* If we let $e(\sigma)$ be the subset of $G_n(\mathbf{R}^k)$ having σ as their Schubert symbold we note that these are spaces whose n-frames, after reducing into Eschelon form, have columns $\sigma_1, \ldots, \sigma_n$ as the pivots; all entries which are not pivot columns can have any real number they wish as the entries, so this subspace is homeomorphic to a Euclidean space (open cell) of dimension $\sum_{i=1}^n (k - (\sigma_i - 1) - (n - i + 1))$. The maximal value of this is $n(k-n)$.

Remark 2.4. Note that this does not rely on any properties of $\mathbf{R}$ other than that $\mathbf{R}^m$ is homeomorphic to an open cell of dimension m. Thus we could have done the exact same analysis for complex Grassmannians in $\mathbf{C}^n$ and we would have obtained a similar CW structure with cells of slightly different dimensions.

We have inclusions $G_n(\mathbf{R}^n) \subseteq G_n(\mathbf{R}^{n+1}) \subseteq \cdots \subseteq G_n(\mathbf{R}^\infty)$, where $\mathbf{R}^\infty = \bigoplus_{i=1}^\infty \mathbf{R}$ induced by the inclusions $\mathbf{R}^n \subseteq \mathbf{R}^{n+1}$ as those vectors with 0 as their last coordinate. Each of these respects the CW structure as defined above, so they are CW inclusions; this produces a CW structure on $G_n(\mathbf{R}^\infty)$.

Definition 2.5. We define $G_n \stackrel{\text{def}}{=} G_n(\mathbf{R}^\infty)$.

Since a Grassmannian is a space encoding information about vector subspaces it comes with a natural definition of a vector bundle.

Definition 2.6. The *universal bundle* γ_{nk} is a bundle $p\colon \gamma_n \longrightarrow G_n(\mathbf{R}^k)$ which has over every point the plane that the point represents. More concretely,

$$\gamma_{nk} = \{(\omega, v) \mid \omega \in G_n(\mathbf{R}^k),\ v \in \omega \subseteq \mathbf{R}^k\} \subseteq G_n(\mathbf{R}^k) \times \mathbf{R}^k.$$

The map $p\colon \gamma_{nk} \longrightarrow G_n(\mathbf{R}^k)$ is projection onto the first coordinate.

When $k = \infty$ we simply write γ_n instead of $\gamma_{n\infty}$.

Example 2.7. Let $n = 1$. Then $G_1(\mathbf{R}^k)$ is $\mathbf{R}P^k$, k-dimensional projective space. We prove that γ_{1k} is nontrivial. Note that a trivial line bundle has a section which is everywhere nonzero; we show that this is not the case for γ_{1k}. Suppose that $s\colon \mathbf{R}P^k \longrightarrow \gamma_{1k}$ is any section, and consider the composition $S^k \longrightarrow \mathbf{R}P^k \longrightarrow \gamma_{1k}$ where the first map is the usual double cover. This takes a point x to a pair $(\{\pm x\}, t(x)x)$ for some continuous $t\colon S^k \longrightarrow \mathbf{R}$. Note that $t(-x) = -t(x)$. Since S^k is connected we must have $t(x_0) = 0$ for some $x_0 \in S^k$.

We finish up this section with an important observation about Grassmannians.

Definition 2.8. Let G be a topological group, and let EG be a weakly contractible space with a free G-action. The space BG is defined to be the quotient of EG by G.

When G is discrete, BG has $\pi_0 BG = *$, $\pi_1 BG = G$ and $\pi_n BG = 0$ for $n > 1$. However, for groups with other topologies this is often not the case. In particular, when $G = O(n)$ it is still an open question what the homotopy groups are.

Theorem 2.9.

$$G_n \simeq BO(n).$$

Proof. As we had for the Grassmannians, we have a sequence of inclusions $V_n(\mathbf{R}^n) \subseteq V_n(\mathbf{R}^{n+1}) \subseteq \cdots \subseteq V_n(\mathbf{R}^\infty)$. As before, we write $V_n \stackrel{\text{def}}{=} V_n(\mathbf{R}^\infty)$. We have the following facts:

- $O(n)$ acts freely on $V_n(\mathbf{R}^k)$.
- $G_n(\mathbf{R}^k)$ is the quotient of $V_n(\mathbf{R}^k)$ by the action of $O(n)$.
- The action respects the inclusions of V_n's and G_n's.

Therefore if we can show that V_n is contractible we will be done.

The map $V_n(\mathbf{R}^k) \longrightarrow S^{k-1}$ given by projecting an n-frame onto its last vector is a fiber bundle with fiber $V_{n-1}(\mathbf{R}^{k-1})$ (where this $\mathbf{R}^{k-1}$ is the hyperplane orthogonal to the last vector). Thus there exists a long exact sequence in homotopy groups

$$\cdots \longrightarrow \pi_{m+1} S^{k-1} \longrightarrow \pi_m V_{n-1}(\mathbf{R}^{k-1}) \longrightarrow \pi_m V_n(\mathbf{R}^k) \longrightarrow \pi_m S^{k-1} \longrightarrow \cdots.$$

Since $\pi_m S^{k-1} = 0$ for $m < k-2$, $\pi_m V_{n-1}(\mathbf{R}^{k-1}) \cong \pi_m V_n(\mathbf{R}^k)$ for such k. By iterating this and taking k large enough we note that

$$\pi_m V_n(\mathbf{R}^k) \cong \pi_m V_1(\mathbf{R}^{k-n+1}) = \pi_m S^{k-n}.$$

Thus for k large enough we can show that $\pi_m(V_n(\mathbf{R}^k)) = 0$.

Now consider $\pi_m V_n$. An element in this group is represented by a homotopy class of maps $S^m \longrightarrow V_n = \bigcup_{k=n}^\infty V_n(\mathbf{R}^k)$. Since S^m is compact this map must factor through the inclusion $V_n(\mathbf{R}^k) \longrightarrow V_n$ for some k.* Assuming that k is large enough (which, since we can increase at will we always can) we know that the map factors as

$$S^m \longrightarrow V_n(\mathbf{R}^k) \longrightarrow V_n;$$

*This fact is actually more complicated than it looks. [Hov99, Proposition 2.4.2] shows that this fact holds for compact spaces assuming that the sequential limit is given by closed inclusions of Hausdorff spaces.

since the first map is null-homotopic, the entire map must be as well. Thus $\pi_m V_n = 0$, as desired. $\square$

Remark 2.10. If we look at complex Grassmannians the same proof works to show that $G_n(\mathbf{C}^\infty) \simeq BU(n)$.

This is just a surprising fact for now, but it will become very important later in our discussion of K-theory.