3. Classification of vector bundles

Reference for this section: [Hat, Section 1.2], [MS74, Section 5].

And now, the big question:

Question. *Given a space B, classify all vector bundles of dimension n over B up to isomorphism.*

For example, when B is a point a vector bundle is a single vector space so any two vector bundles with the same dimension are isomorphic.

Before we begin our "classification" (and I warn you that this answer will not be completely satisfying as it will be pretty much uncomputable) we need to discuss another way of constructing vector bundles.

Definition 3.1. Let $p\colon E \longrightarrow B$ be a vector bundle, and let $f\colon B' \longrightarrow B$ be any map. The *pullback bundle* $p'\colon f^*E \longrightarrow B'$ is defined in the following manner. The space f^*E is defined by

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = p(e)\},$$

topologized as a subspace of $B' \times E$. The map to B' is projection onto the first coordinate. The vector space structures on the fibers come from the vector space structures on the fibers of E over B.

Note that if E and E' are isomorphic as bundles over B then f^*E and f^*E' are isomorphic as bundles over B'. In addition, we have a commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{f'} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

where p and p' are bundle maps. From the definition of f^*E this is a pullback square, and in fact $p'\colon f^*E \longrightarrow B'$ can simply be defined as the pullback of p along f; in that case, we would need to check that it is actually a vector bundle. Note that f' takes every fiber in f^*E isomorphically to a fiber in E.

Remark 3.2. This same construction works to pull back fiber bundles; the pullback of a fiber bundle with fiber F is another fiber bundle with fiber F.

The big classification theorem for vector bundles is the following:

Theorem 3.3. *Suppose that B is paracompact. Let $\mathrm{Vect}_n(B)$ be the set of isomorphism classes of n-dimensional vector bundles over B. Then the map*

$$[B, G_n] \longrightarrow \mathrm{Vect}_n(B) \qquad \textit{given by} \qquad f \longmapsto f^*\gamma_n$$

is a bijection.

This is a very nifty result: it says that vector bundles *up to isomorphism* as the same as *homotopy classes of maps into Grassmannians*. This is the first indication that homotopical invariants contain a lot of information about *geometry*. Because of this theorem γ_n is sometimes called the *universal bundle.*

The proof of this theorem is quite long, so we break it up into several intermediate results. We begin with a quick note about paracompactness.

Definition 3.4. A space is *paracompact* if every open cover has a refinement which contains a locally finite subcover.

Paracompactness is a much less restrictive condition to impose than compactness, since all CW complexes are paracompact (take the interiors of all the cells). In addition, we have already seen a very natural paracompact space come up in our discussion: G_n is paracompact, since (even without knowing that it is a CW complex) we note that it is a sequential union of compact Hausdorff spaces.

An important result about paracompact spaces:

Lemma 3.5 ([Hat, Lemma 1.21])**.** *Given any open cover $\{U_\alpha\}$ of a paracompact space X, there exists a countable open cover $\{V_i\}$ such that the following conditions hold:*

(1) *For each i we can write $V_i = \amalg U'_\alpha$ with each U'_α an open (possibly empty) subset of U_α.*
(2) *There exists a partition of unity $\{\varphi_i\}$ subordinate to $\{V_i\}$.*

Let us now turn our attention to the proof of Theorem 3.3. We begin by checking that the given map is well-defined.

Lemma 3.6. *If $f, g\colon X \longrightarrow Y$ are homotopic and $E \longrightarrow Y$ is a vector bundle over Y then f^*E and g^*E are isomorphic.*

Proof. We follow [Hat, Proof of Proposition 1.7].

Let $H\colon X \times I \longrightarrow Y$ be a homotopy from f to g, and consider H^*E. This contains f^*E as the restriction of the bundle to $X \times \{0\}$ and g^*E as the restriction of the bundle to $X \times \{1\}$, so it suffices to check that for *any* map $h\colon X \times I \longrightarrow Y$, the restrictions of h^*E to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.

First, let us consider this when h^*E is trivial. In that case we have a trivialization $\tau\colon h^*E \longrightarrow X \times I \times \mathbf{R}^n$; the restrictions of this trivialization to the preimage of $X \times \{0\}$ and $X \times \{1\}$ are both homeomorphisms to $X \times \{*\} \times \mathbf{R}^n$. The composition of one of these with the inverse of the other one gives the desired isomorphism.

Let $f\colon X \longrightarrow [0,1]$ be any map; let X_f be the graph of f inside $X \times I$, and let E_f be the restriction of E to X_f. The argument above shows that given two functions $f, g\colon X \longrightarrow [0,1]$, E_f is isomorphic to E_g.

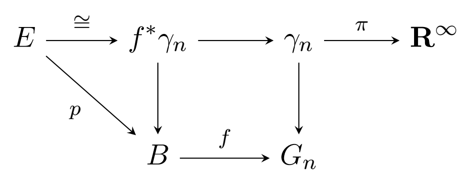
Now let us consider even more generality. Suppose that we are given two functions $f, g\colon X \longrightarrow [0,1]$ which are equal outside of a closed set V. We are also given a trivialization $\tau\colon U \times I \longrightarrow X \times I \times \mathbf{R}^n$ of h^*E above $U \times I$, where U is an open set containing V. In this case we also have that $E_f \cong E_g$, by the same argument.

It turns out that this is sufficient, after a bit of massage, to prove the result. Suppose that we have a cover $\{U_\alpha\}$ of X such that h^*E is trivial above $U_\alpha \times I$ for all α. Use Lemma 3.5 to produce a countable cover $\{V_i\}$ with a subordinate partition of unity $\{\varphi_i\}$. Note that for all i, h^*E is trivial over $V_i \times I$. Let $\psi_i = \sum_{i=0}^{i} \varphi_i$, so that $\psi_0 = 0$ and $\psi_\infty = 1$. Since ψ_i is equal to ψ_{i-1} outside of the support of φ_i, we are in the case we just discussed and $E_{\psi_i} \cong E_{\psi_{i-1}}$. This isomorphism is the identity outside of the support of φ_i, so we can compose all of these to get an isomorphism $E_{\psi_\infty} \cong E_{\psi_0}$, which is exactly the desired isomorphism. $\square$

Exercise. Prove that for any paracompact X and any bundle $E \longrightarrow X \times I$ there exists an open cover $\{U_\alpha\}$ of X such that E is trivial over $U_\alpha \times I$.

Lemma 3.7. *For any vector bundle $p\colon E \longrightarrow B$, an isomorphism $E \cong f^*\gamma_n$ is equivalent to a map $g\colon E \longrightarrow \mathbf{R}^\infty$ which is a linear injection on each fiber.*

Proof. Suppose that we have a map $f\colon B \longrightarrow G_n$ and an isomorphism $E \cong f^*\gamma_n$. Then we have the diagram



We let g be the composition across the top; since each of the maps in the composition is a linear injection on each fiber, so is the composition.

Now suppose that we have such a map $g\colon E \longrightarrow \mathbf{R}^\infty$. We let $f(b) = g(p^{-1}(b))$. This produces a diagram as above by mapping a point $e \in E$ to $g(e)$ inside $f^*\gamma_n$. $\square$

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. We follow [Hat, Theorem 1.16].

First we consider injectivity. Suppose that $E \cong f_0^*\gamma_n$ and $E \cong f_1^*\gamma_n$. These produce maps $g_0, g_1\colon E \longrightarrow \mathbf{R}^\infty$ which are linear injections on fibers. We claim that g_0 and g_1 are homotopic. Given a homotopy $G\colon E \times I \longrightarrow \mathbf{R}^\infty$ we can define a homotopy $F\colon B \times I \longrightarrow G_n$ by $F(b,t) = G(p^{-1}(x), t)$. Thus it just remains to construct G.

It is tempting to write $G(e,t) = tg_0(e) + (1-t)g_1(e)$. This is clearly well-defined and continuous; however, it may not be a linear injection on some fibers. Note, however, that whenever $g_0(p^{-1}(b))$ and $g_1(p^{-1}(b))$ are linearly independent, then $G|_{p^{-1}(B)\times I}$ is a linear injection. Let $L^o\colon \mathbf{R}^\infty \times I \longrightarrow \mathbf{R}^\infty$ be the map defined by

$$L^o((x_1,\ldots),t) = t(x_1,\ldots) + (1-t)(x_1,0,x_2,0,\ldots).$$

Thus L moves $\mathbf{R}^\infty$ to the odd coordinates; similarly, we can define a homotopy L^e which moves $\mathbf{R}^\infty$ to the even coordinates. Then we can define

$$G(e,t) = \begin{cases} L^o(g_0(e), 3t) & \text{if } t \le \frac{1}{3}, \\ (2-3t)L^o(g_0(e),1) + (3t-1)L^e(g_1(e),1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ L^e(g_1(e), 3-3t) & \text{if } t \ge \frac{2}{3}. \end{cases}$$

Thus g_0 and g_1 are homotopic, so so are f_0 and f_1.

Now we turn to surjectivity. Suppose that $p\colon E \longrightarrow B$ is a vector bundle. If we can construct a map $g\colon E \longrightarrow \mathbf{R}^\infty$ as above we will be done. We do this, as before, by restricting to trivial open sets and then extending using paracompactness and a partition of unity. First, suppose that E is trivial over U. On the set $U \times \mathbf{R}^n$ we can define a map to $\mathbf{R}^n$ by simply projecting to the second variable. Now take a countable cover $\{U_i\}_{i=1}^\infty$ with a subordinate pertition of unity $\{\varphi_i\}$ such that E is trivial over all U_i. Let $pr_i\colon U_i \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be the projection onto the second coordinate; we can extend this to a map $g_i\colon E \longrightarrow \mathbf{R}^n$ by setting it to be $\varphi_i(p(e))pr_\alpha(e)$ when $e \in U_i \times \mathbf{R}^n$ and 0 otherwise. When φ_i is nonzero this is a linear injection. We then define

$$g(e) = (g_1(e), g_2(e), \ldots) \in (\mathbf{R}^n)^\infty \cong \mathbf{R}^\infty.$$

Since at every point at least one φ_i is nonzero, this is a linear injection on each variable, as desired. $\square$

Motivated by this theorem we introduce the following definition:

Definition 3.8. A *classifying map* of a vector bundle $E \longrightarrow B$ is a homotopy class of maps $[f]\colon B \longrightarrow G_n$ such that $E \cong f^*E$.

The upshot of this theorem is that G_n knows everything about the structure of vector bundles. Let's explore what this means in an example.

Example 3.9. Let E, E' be two vector bundles, of ranks m and n, over B. We can form another vector bundle, called their *Whitney sum*, by taking fiberwise direct sums. This is often written $E \oplus E'$. In other words, we can form the pullback

$$\begin{array}{ccc} E \oplus E' & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p'} \\ E' & \xrightarrow{p} & B \end{array}$$

and consider $E \oplus E'$ as a bundle over B. If we know the classifying maps of E and E', what is the classifying map of $E \oplus E'$?

Consider the following map $G_m \times G_n \longrightarrow G_{m+n}$. A point (ν, ν') in $G_m \times G_n$ is a pair of subspaces of $\mathbf{R}^\infty$. Now consider two copies of $\mathbf{R}^\infty$, one sitting as the odd coordinates of $\mathbf{R}^\infty$ and one sitting as the even coordinates. We can consider ν as a subspace of this first $\mathbf{R}^\infty$ and and ν' as a subspace of the second and take the $m+n$-subspace spanned by the two. This gives us a point in G_{m+n}. We'll call this map $\oplus$.

Let $f\colon B \longrightarrow G_m$ and $f'\colon B \longrightarrow G_n$ be classifying maps for E and E', respectively. Consider the map

$$f_\oplus\colon B \xrightarrow{\Delta} B \times B \xrightarrow{f \times f'} G_m \times G_n \xrightarrow{\oplus} G_{m+n}.$$

We claim that $f_\oplus^* \gamma_n$ is isomorphic to $E \oplus E'$. (Exercise: prove this!)

We can perform similar things with classifying maps for tensor products, skew-symmetric products, dual spaces, etc.

In a perfect world we would now be able to classify all vector bundles on B by computing $[B, G_n]$, and this computation would be effective enough that we could use it to, for example, determine when two vector bundles are isomorphic. Unfortunately we do not live in such a world, and in general such computations are incredibly difficult. Thus we now begin the search for a more computable invariant of vector bundles.

Remark 3.10. There are further results along these lines for other structured bundles. For example, for any topological group G, the space BG classifies the principal G-bundles (the bundles whose fibers are G, together with continuous G-action).