PROOF BY CONTRADICTION

1. Exercises

(1) Prove that there are infinitely many prime numbers.
(2) Prove that $\sqrt{2}$ is irrational.
(3) Prove that there is no polynomial of degree at least 1 with integer coefficients such that all of the values $P(0), P(1), P(2), \ldots$ are prime numbers.
(4) Suppose that $F = \{E_1, \ldots, E_s\}$ is a family of subsets with r elements of some set X. Show that if the intersection of any $r + 1$ (not necessarily distinct) sets in F is nonempty then the intersection of all of the sets in F is nonempty.
(5) Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

2. Problems

(1) The polynomial $p(x)$ has integer coeffient. The leading coefficient, the constant term and $p(1)$ are all odd. Prove that $p(x)$ has no rational roots.
(2) Show that no set of nine consecutive integers can be partitioned into two sets with the product of the elements of the first set equal to the product of the elements of the second set.
(3) Every point of three-dimensional space is colored red, green, or blue. Prove that for one of the colors it is the case that for all real $x > 0$ there exist two points a distance x apart that are that color. (Note: this FIXES the color first!)
(4) Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying
$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$
for all positive integers x and y.
(5) Show that the interval $[0, 1]$ can't be partitioned into two disjoint sets A and B such that $B = A + a$ for some real a.
(6) Show that a graph with $2n$ points and $n^2 + 1$ edges must contain a triangle, but we can find a graph with $2n$ points and n^2 edges that does not contain a triangle.
(7) S is an infinite set of points in the plane. The distance between any two points of S is an integer. Show that S is a subset of a line.