FIRST-YEAR PRIZE EXAM

TIME LIMIT: 2 HOURS

Please write all solutions in the answer booklet. You must justify all answers. Partial credit will be assigned liberally, so please submit even incomplete solutions.

Problem 1. The planet of Graphia has cities connected by roads. All roads are bidirectional, and no two roads cross except at cities. The spaces in between roads are fields where the planet grows its food. The government has decided that all roads should be one-way and would like to assign a direction to each road. However, farmers need to be able to ride around their fields to check on how the crops are doing. Prove that if every city has an even number of roads leading to it then the government can successfully carry out their scheme.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Write $f^{(n)}$ for the nth derivative of f. Suppose that for all **integers** n,

$$f\left(\frac{1}{n}\right) = \frac{1}{1+n^4}.$$

Find $f^{(n)}(0)$ for all n.

Problem 3. Show that

$$\lim_{y\to 0} \pi y \int_{-\infty}^{\infty} \frac{f(x)}{x^2+y^2}\,dx = f(0)$$

for all continuous, bounded, continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

Problem 4. You are given a line of N identical coins placed either `H`eads of `T`ails up. For every coin that shows `H`eads you count how many coins show `T`ails to the left of it. Summing up all the numbers thus obtained one gets 136. Then for every coin that shows `T`ails you count how many coins show `H`eads to the left of it. Adding up all of these numbers you get 153. If the first two coins are `HT` find N.
Example: if the row of coins were `HTHHTTH` the two numbers obtained would be $0 + 1 + 1 + 3 = 5$ and $1 + 3 + 3 = 7$