FIRST- AND SECOND-YEAR PRIZE EXAM

TIME LIMIT: 2 HOURS

You may not use textbooks, notes, calculators, the internet, etc. Pencil, pen, and paper are all of the aids that are allowed.
Please write each solution on a separate page and submit scans separately to Gradescope.
You must justify all answers. Partial credit will be assigned liberally, so please submit even incomplete solutions.

Problem 1. Let C be a collection of 28 distinct points in $\mathbb{R}^3$ whose coordinates are integers. Show that there exist two points $p_1, p_2 \in C$ such that the line segment connecting p_1 to p_2 contains at least two points with integer coordinates in its interior.

Problem 2. Let ABC be an isosceles triangle with $AB = AC = 4$. Point D is on side AB (different from both A and B) and point E is on AC with $AD = DE = EC$. If the length of AD is an integer, what could the length of BC be? (You must find all possible lengths for full credit.)

Problem 3. Suppose that we have a sphere in $\mathbb{R}^3$. What are the possible numbers of rational points on the sphere?

Problem 4. A country has some number of cities. Each pair of cities is directly connected by exactly one type of public transportation: plane, train, or bus. No city has all three kinds of connections, and no three cities are all linked by the same type of transportation. (Thus, for example, if city A has a bus terminal and an airport, it cannot have a train station. If A is connected to both B and C via bus, then B and C are not linked by bus.) If we know that this country uses all three of plane, train, and bus, connections, what is the maximal number of cities in the country?