GEOMETRIC TRANSFORMATIONS IN OLYMPIADS

1. Some types of geometric transformations

isometries: are transformations that preserve all distances. Isometries preserve lengths, areas, and angles. All translations are compositions of reflections, translations and rotations.

homotheties: scale the plane by a constant. Homotheties preserve angles and ratios of lengths and areas.

affine transformations: are compositions of translations and linear transformations of the plane. They preserve ratios of lengths along a line and ratios of areas.

spiral similarities: are compositions of rotations and homotheties.

2. Warm-up problems

(1) Show that given any two triangles ABC and $A'B'C'$ there exists an affine transformation that takes A to A', B to B' and C to C'. Use this to show that the three medians of a triangle are concurrent (intersect at a point).

(2) Given four points A, B, C, D in the plane such that $ABCD$ is not a parallelogram, show that there exists a unique spiral similarity that sends A to B and C to D.
(Hint: First show that if a spiral similarity exists then it is unique. Now let X be the intersection of lines AB and CD. Let ω_1 and ω_2 be the circumcircles of ACX and BDX. If Y is the second intersection of ω_1 and ω_2, show that Y is the center of the spiral similarity.)

(3) (a) Show that the three altitudes of a triangle are concurrent.
(Hint: First, show that the three perpendicular bisectors of a triangle are concurrent. Now draw a line parallel to the opposite side through each vertex of the triangle; what is the relation of the smaller triangle and the larger one you just drew?)

(b) Let ABC be a triangle which is not equilateral. Let G be the centroid (intersection of medians) of ABC, let O be the circumcenter of ABC, and let H be the orthocenter (intersection of the altitudes). Show that O, G and H are collinear (in that order!) and that $|GO| = 2|GH|$.

3. More problems

(1) Let A be the area of the region of the plane in the first quadrant bounded by the x-axis, the line $y = \frac{1}{2}x$ and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the y-axis, the line $y = mx$ and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

(2) What is the ellipse of largest area that can be inscribed in a 3-4-5 right triangle?

(3) A straight line cuts the asymptotes of a hyperbola at A and B and the hyperbola itself at P and Q. Show that $AP = BQ$.

(4) Three lines passing through an interior point of a triangle parallel to the sides determine three triangles and three parallelograms. If S is the area of the original triangle and S_1, S_2, S_3 are the areas of the newly formed triangles, show that $S_1 + S_2 + S_3 \geq \frac{1}{3}S$.

(5) Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines in the plane. Let C_{ijk} be the circumcircle of the triangle formed by lines ℓ_i, ℓ_j, ℓ_k. Prove that all four of these circles share a common point.

(6) A block of wood has the shape of a right circular cylinder with radius 6 and height 8, and its entire surface has been painted blue. Points A and B are chosen on the edge of one of the circular faces of the cylinder so that arc AB on that face measures $120°$. The block is then sliced in half along the plane that passes through point A, point B, and the center of the cylinder, revealing a flat, unpainted face on each half. Find the area of one of these unpainted faces.

(7) Triangle ABC has an area of 1. Points E, F, G lie on sides BC, CA, AB such that AE bisects BF at R, BF bisects CG at S and CG bisects AE at T. Find the area of $\triangle RST$.

(8) Let $ABCD$ be inscribed in a circle. Lines AB and CD intersect at P; lines AD and BC intersect at Q. Let QE and QF be the tangents to the circle, where E and F are the tangency points. Prove that E, F, P are collinear.[1]

[1]The nice transformational solution requires some projective geometry.