GRAPH THEORY

A graph $G = (V, E)$ is an ordered pair where V is the *vertex set* of the grpah, and E is the *edge set*. An edge consists of an unordered pair (v, v') of vertices; if the graph is *directed* it is an ordered pair. Some graphs allow multiple edges between vertices. The *degree* of a vertex is the number of edges containing that vertex. If the graph is directed, the *in-degree* is the number of edges whose second coordinate is the vertex; the *out-degree* is the number of edges whose first coordinate is the vertex.

1. Warm-up problems

(1) Prove that in an undirected graph, $\sum_{v \in V} \deg(v) = 2\#E$. Prove that in a directed graph, $\sum_{v \in V} \text{in-deg}(V) - \text{out-deg}(V) = 0$.
(2) How many edges must a graph with n vertices have to guarantee that it is connected?
(3) An *Eulerian path* in a graph is a path through the graph that travels along each edge exactly once.
   (a) When does a graph contain an Eulerian path? When is this path a cycle (starts and ends at the same vertex)?
   (b) Is it possible for a knight to travel through a chessboard in such a way that it makes every possible move exactly once? (We consider doing a move and its reverse the same thing.)
(4) An graph is *bipartite* if we can write $V = A \cup B$ with $A \cap B = \emptyset$ in such a way that no edge connects two vertices of A or two vertices of B. For any vertex $v \in V$, we write $N(v) = \{v' \in V \mid (v, v') \in E\}$.
   (a) Show that a graph is bipartite if and only if it has no odd cycles.
   (b) Suppose that in a bipartite graph there exists a subset of edges $E' \subseteq E$ such that each vertex of A appears as the endpoint of exactly one edge of E', and such that each vertex of B appears as the endpoint of at most one edge of E'. Show that for any subset $A' \subseteq A$,

$$\left| \bigcup_{v \in A'} N(v) \right| \geq |A'|.$$

   (c) Prove the converse of the previous part. (Hint: suppose that E' does not exist, and let E'' be the maximal subset of E such that each vertex of V is the endpoint of at most one edge in E''. Let $u \in A$ be a vertex which is not an endpoint of an edge in E'', and consider all maximal-length alternating paths starting at u; here, an alternating path alternates edges not in E'' with edges in E''. Show that all such paths must end in A and find a contradiction.)

2. More problems

(1) Consider a chessboard with the two corner black squares removed. Can the remaining squares be tiled with dominoes? (A domino covers two adjascent squares.)
(2) Some people play a round-robin chess tournament. Prove that it is always possible to order them in such a way that the first person beat the second, who beat the third, and so on. Is this ranking necessarily unique?
(3) The country of Euleria has $2n$ cities connected by unpaved roads. It is possible to get from any city to any other city along these roads. Prove that it is possible to pave some of the roads so that each city has an odd number of paved roads leading to it.
(4) Suppose that G is a connected graph with k edges. Prove that it is possible to label the edges $1, \ldots, k$ in such a way that for each vertex of degree greater than 1, the greatest common divisor of the edges at that vertex is 1.
(5) A standard deck of cards is dealt into 13 columns of 4 cards each. Prove that there is a way to choose one card from each column to get a card of each rank.
(6) A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, with each team playing one other team on each day. In each game, one of the teams won. Is it possible to select a winning team from each day in such a way that no team is selected twice?

(7) Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in an $n \times n$ square so that the k-th row, from left to right, reads $(k-1)n+1, \ldots, kn$. Color the grid so that half of the squares in each row and half of the squares in each column are black, and the rest white. Prove that the sum of the numbers in the white squares is the same as the sum of the numbers in the black squares.