INDUCTION

Induction is a great proof technique whenever you can build solutions for large cases using small cases as building blocks. The classic description of how to do a proof by induction is the following sequence of two steps:

(A) Prove the case for the smallest n, usually $n = 1$.
(B) Assume that the desired statement holds for $n = 1, \ldots, k-1$; now prove it for $n = k$.

1. Exercises

(1) Let x be a real number such that $x + 1/x$ is an integer. Prove that $x^n + 1/x^n$ is an integer for all n.

(2) Prove the following formulas by induction:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

where in the last line, F_k is the k-th Fibonnaci number. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. (Such a definition is called a *recursion*; these work very well for inductive proofs.)

(3) Prove that for all n,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

(4) (Base Fibonacci) Prove that for every integer n there exists a unique sequence $i_2, i_3, \ldots$ such that (a) $i_k = 0, 1$ for all k, (b) $i_k i_{k+1} = 0$ for all $k \geq 2$, and (c) $\sum_{k=2}^{\infty} i_k F_k = n$.

(5) One side of a street with n houses has gotten together and decorated all of the houses with Christmas lights. However, the light bothered the people living on the other side of the street. The people living on the street agreed that some of the houses could be lit up each night, but no two houses next to one another can be lit up. In how many ways can the street be lit up?

(6) Write the numbers $1, 1$ on the board. Now write their sum between them to get $1, 2, 1$. Repeat again to get $1, 3, 2, 3, 1$. Continue in this way until you have done this 100 times. What is the sum of the numbers on the board?

2. Problems

(1) You are given two rational numbers, s and t. When is it possible to construct a sequence

$$s_0, s_1, \ldots, s_n$$

such that $s_0 = s$, $s_n = t$ and for all $i = 1, \ldots, n$ we have $s_i = 1/s_{i-1}$ or $s_i = s_{i-1} + 1$?

(2) Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n}}.$$

(3) An $n \times n$ matrix is called *good* if for each $i = 1, \ldots, n$, the union of the i-th row and i-th column contain all of the integers 1 through $2n - 1$. Prove that there is no good matrix with $n = 2017$, but good matrices exist for infinitely many values of n.

(4) The numbers $1, 2, \ldots, 2n$ are arbitrarily divided into two sets, $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$; we assume that

$$a_1 < a_2 < \cdots < a_n \qquad \text{and} \qquad b_1 > b_2 > \cdots > b_n.$$

Find

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|.$$

(5) Given $T_1 = 2$, $T_{n+1} = T_n^2 - T_n + 1$ for $n > 1$, show that T_n and T_m are relatively prime for all $n \neq m$. (Two numbers are relatively prime if they share no prime factors.)

(6) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.

(7) Let $g(x, y)$ be a continuous function $[0, 1] \times [0, 1] \to \mathbb{R}$. Suppose that for all x, y we have

$$g(x, y) = \int_0^x \int_0^y g(u, v)\, du dv.$$

Prove that $g(x, y) = 0$ for all x, y.

(8) Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2}$$

for any x_1, x_2. Prove that

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

for any n and any $x_1, \ldots, x_n$.

(9) Let $a_1, a_2, \ldots$ be a sequence of distinct positive integers. Prove that for any positive integer n,

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n + 1}{3}(a_1 + a_2 + \cdots + a_n).$$