NUMBER THEORY

An integer $p \geq 2$ is *prime* if it has no divisors other than 1 and itself. A number which is not prime is *composite*.

For any integer $n \geq 1$ we can do *modular arithmetic modulo* n, by which we mean that we work with numbers as though their remainders upon dividing by n is all that is important. Addition and multiplication work modulo n just like it does for ordinary integers. (For example, consider that the last digit of the sum of two numbers is sum of the last digits of the two numbers, where if the sum is larger than 10 we again take its last digit. It works similarly for multiplication.)

(1) Which of the numbers $1, 101, 10101, 1010101, \ldots$ are prime?

(2) Let $d_1 \ldots d_9$ be a 9-digit number. Define the digit e_i to have the property that $d_1 \ldots e_i \ldots d_9$ is divisible by 7; this gives us a new 9-digit number $e_1 \ldots e_9$. (For example, if we start with 123456789 then $e_5 = 3$.) Now do the same process again to this new number to produce the number $f_1 \ldots f_9$. Prove that for all $i = 1, \ldots, 9$, $d_i - f_i$ is divisible by 7.

(3) Let $a_0 = 1$, $a_1 = 2$ and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2019}.

(4) Let f be a nonconstant polynomial with positive integer coefficients. Show that if n is a positive integer then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

(5) Suppose that $0 \leq a, b, c, d \leq 99$ are integers with $a \leq b$ and $c \leq d$. Let $n_i = 101i + 1002^i$. Prove that if $n_a + n_b = n_c + n_d \pmod{10100}$ then $a = c$ and $b = d$.

(6) Let p be an odd prime and let

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$

Prove that if $0 \leq a < b \leq p-1$ then $F(b) - F(a)$ is not divisible by p.

(7) Let h and k be positive integers. Prove that for every $\epsilon > 0$ there exist m and n such that

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon.$$

(8) Let T be the set of all triples of integers (a, b, c) such that there exists a triangle with side lengths a, b and c. Find the sum

$$\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c}$$

in lowest terms.

(9) For which positive integers n does there exist an $n \times n$ matrix with integer entries such that the dot product of any row with itself is even but the dot product of two different rows is odd?

(10) Let q be an odd positive integer, and let N_q denote the number of integers a such that $0 < a < q/4$ and $\gcd(a, q) = 1$. Show that N_q is odd if and only if q is of the form p^k with k a positive integer and p a prime congruent to 5 or 7 modulo 8.

(11) For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to n, which satisfy

$$(x^2 + y^2)^m = (xy)^n.$$