The Pigeonhole Principle. *If $kn+1$ objects are distibuted among n holes then at least one hole will have $k+1$ objects in it.*

1. Exercises

(1) Suppose $a_1, \ldots, a_n \geq 0$ are real numbers with average greater than A. Prove that for some i, $a_i > A$.
(2) Inside of a circle of radius 4 are chosen 61 points. Prove that there are at least two points a distance at most $\sqrt{2}$ apart.
(3) Prove that every set of 10 two-digit numbers has two disjoint subsets with the same sum of elements.
(4) Given nine points inside the unit square prove that some three of them form a triangle whose area does not exceed $\frac{1}{8}$.
(5) Each of nine lines divides a square into two quadrilaters with the ratio of their areas equal to $r > 0$. Prove that at least three of the lines are concurrent.
(6) Given a set M of 2018 distinct positive integers, none of which has a prime divisor greater than 26, prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.
(7) Let a_j, b_j, c_j be integers for $1 \leq n \leq N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values for j.

2. Problems

(1) Forty-one rooks are placed on a 10×10 chessboard. Prove that there exists a set of 5 rooks none of which attack each other. (Recall that rooks attack each piece in their same row or column.)
(2) A sequence of m positive integers contains exactly n distinct terms. Prove that if $2^n < m$ then there exists a block of consecutive terms whose product is a perfect square.
(3) Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$ and $A+4B$ are all invertible matricse whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.
(4) Let $x_1, \ldots, x_{20}$ be positive integers each of which is less than or equal to 18. Let $y_1, \ldots, y_{18}$ be positive integers each of which is less than or equal to 20. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.
(5) Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq \frac{5}{2}$.
(6) Prove that for every set $X = \{x_1, \ldots, x_n\}$ of n real numbers, there exists a nonempty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

(7) Let $x_1, \ldots, x_k$ be real numbers such that the set

$$A = \{\cos(n\pi x_1) + \cos(n\pi x_2) + \cdots + \cos(n\pi x_n) \,|\, n \geq 1\}$$

is finite. Prove that all of the x_i are rational numbers.
(8) Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \ldots, \pm 1)$ in n dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

(9) Let S be a region in the plane (not necessarily convex) with area greater than the positive integer n. Show that it is possible to translate S (i.e., slide without turning or distorting) so that S covers at least $n+1$ lattice points.

(10) Let n be a positive integer. Choose any $n+1$-element subset of $\{1, 2, \ldots, 2n\}$. Show that this subset must contain two integers, one of which divides the other.