SYMMETRY: EXPLOITING AND BREAKING

1. Warmup

(1) Compute $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ in your head.
Hint: Draw the graphs of $\cos^2 x$ and $\sin^2 x$ from 0 to $\frac{\pi}{2}$. How are the areas below them related?

(2) Find the length of the shortest path from the point $(3,5)$ to the point $(8,2)$ that touches the x-axis and also touches the y-axis.
Hint: Suppose that the path only needed to touch the x-axis. Reflect the part of the path before it touches the x-axis across the x-axis, so that now it's a path from $(3,-5)$ to $(8,2)$. How are the lengths of these paths related?

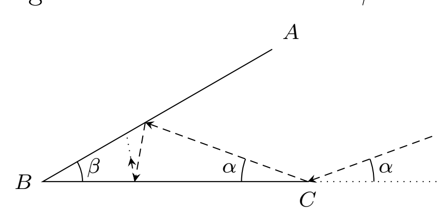
(3) S is a set of n points in the plane. It also has the property that given any two $A, B \in S$, there exists a $C \in S$ such that $\triangle ABC$ is equilateral. What is the maximal value of n?
Hint: Choose A, B, C so that the area of $\triangle ABC$ is maximal. Where can other points in S be located relative to A, B and C?

(4) Let $f(x)$ be a polynomial which is strictly positive for all real x. Let $g(x) = \sum_{n=0}^{\infty} f^{(n)}(x)$, where $f^{(n)}(x)$ is the n-th derivative of f. Prove that $g(x)$ is also strictly positive for all real x.
Hint: Explain why $g(x)$ has a minimum value. What happens at the minimum of g?

(5) Given an infinite chessboard that contains a positive integer in each square. If the value in each square is the average of the four neighbors to the north, south, east and west, prove that all the values in the squares are equal.
Hint: What happens at the square with the minimal value? (Extra credit: if we remove the "positive" requirement, can you construct an example where not all squares have the same value?)

2. Problems

(1) A billiard ball strikes ray $\overrightarrow{BC}$ at point C, with angle of incidence α. The billiard ball continues on its path, bouncing off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule: angle of incidence equals angle of reflection. If $AB = BC$, determine the number of times the ball will bounce off the two segments in terms of α and β.



(2) Compute

$$\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}.$$

(3) Given a point (a,b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a,b), one on the x-axis and one on the line $y = x$. (You may assume such a minimal triangle exists.)

(4) Prove the reflection property of the ellipse: if a pool table is built with an elliptical wall and you shoot a ball from one focus to any point on the wall, the ball will reflect off the wall and travel to the other focus.
(5) Four points are chosen at random on a sphere. What is the probabily that the center of the sphere lies inside the tetrahedron whose vertices are at the four points?
(6) Prove that you can tile the plane using infinitely many squares and infinitely many equilateral triangles. Now prove that it is impossible to tile the plane using infinitely many squares and only finitely many equilateral triangles.
(7) A *palindrome* is a number or word that is the same when read forwards or backwards, such as "racecar" or "1234321". Can the number obtained by writing the numbers from 1 to n be a palindrome if $n > 1$?