Research Statement – Matthew C. B. Zaremsky

I study geometric group theory, which means that I use geometric and topological tools to understand properties of infinite groups. In a typical problem, there is an infinite group acting nicely on a geometric or topological space, and one wishes to extract information about the group from information about the space. For example, one might want to know if the group is finitely generated, finitely presented, or possesses even stronger *finiteness properties*. One main technique I employ in my research is *combinatorial Morse theory*, which is a tool for turning difficult *global* topological problems into easier *local* ones.

Aspects of my research build off of seminal work by Brown [Bro87], Quillen [Qui78], Bieri–Neumann–Strebel [BNS87] and Bieri–Renz [BR88] on topological properties of infinite groups, and by Bestvina and Brady [BB97] on combinatorial Morse theory. Many of my research projects have involved using Morse theory to solve open problems that had proved too difficult for other methods; see for example Theorems 2.1, 2.2, 3.2, 3.3 and 4.2 below.

Geometric group theory is a comparatively new, fast-moving and broad field of study, encompassing many diverse specialties. It is also particularly well suited to involving students in research. In the summer of 2016 I ran an undergraduate research project with my student Eidan Maimoni, working on constructing new examples of groups with interesting properties. Also, recent work of mine with Stefan Witzel [WZ16b] was used by a team of undergraduates, in a summer REU at Miami University working under Dan Farley, to produce potential counterexamples to an interesting conjecture about *Thompson's group* V [BZFGM14].

In the coming sections I focus on five particular aspects of my research. In Section 1, I define *classifying spaces* of groups, and discuss my work involving *configuration spaces*. In Sections 2 and 3, which encompass my most major results, I discuss, respectively, *finiteness properties* of groups, and topological properties *at infinity*. Sections 4 and 5 focus respectively on *homological stability* for families of groups and groups acting on *buildings*. In all these projects, my work involves using geometric and topological tools to deduce important properties of interesting groups, and build connections among the fields of algebra, geometry and topology.

1. Classifying spaces

A connected CW-complex X is called a *classifying space* for a group G if the fundamental group $\pi_1(X)$ is isomorphic to G, and the higher homotopy groups $\pi_k(X)$ are trivial for all $k \geq 2$. Classifying spaces are important for many reasons, for example the homology of a group equals the homology of its classifying space, which is in theory easier to compute.

Braid groups are examples of groups with very nice classifying spaces. The n-strand braid group can be defined via its *standard presentation*

$$B_n := \langle s_1, \dots, s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i, \text{ and } s_i s_j = s_j s_i \text{ for } |i-j| > 1 \rangle.$$

Visually, an element of B_n is a picture of n strands winding around each other (Figure 1).

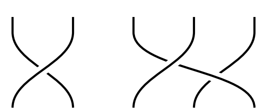

FIGURE 1. An element of the braid group B_5.

Looking at horizontal cross-sections from a bird's-eye view, such a picture describes n distinct points moving around in the plane $\mathbb{C}$. This intuition translates to a classifying space for B_n, namely, the *configuration space* $\mathcal{C}B_n = \{\{z_1, \dots, z_n\} \subseteq \mathbb{C} \mid z_i \neq z_j \text{ for } i \neq j\}$, of n points in $\mathbb{C}$.

It turns out *Thompson's group* F has a similar, nice classifying space $\mathcal{C}F$ of configurations. The group F is the group of piecewise linear, orientation-preserving, continuous bijections $f\colon [0,1] \to [0,1]$ of the unit interval, such that the slopes of the linear pieces are all powers of 2, and all the points of non-differentiability lie in $\mathbb{Z}[\frac{1}{2}]$. There are many reasons that people are interested in F, for instance it was the first example of a torsion-free group of type F_∞ (defined below) that contains copies of $\mathbb{Z}^n$ for all n. The elements of F can be encoded into *strand diagrams* that look similar to braid diagrams, except instead of braiding, the strands *split* and *merge* (Figure 2).

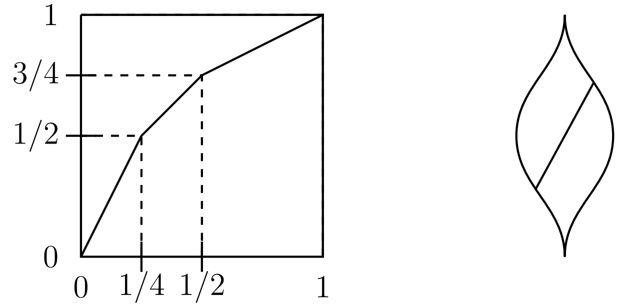

Figure 2. An element of Thompson's group F, as a function $[0,1] \to [0,1]$ with two breakpoints in the domain and two in the range, and as a strand diagram with two splits and two merges.

In his 2004 PhD thesis, Belk proposed a classifying space for F, namely the space $\mathcal{C}F$ of configurations of any number of points in the real line $\mathbb{R}$ such that two points *may* coincide, but no three points may be within a ball of radius 1. One can see the idea by taking horizontal cross-sections of strand diagrams. An idea of a proof was sketched in Belk's thesis, but turning this into a rigorous proof seemed difficult. In joint work with Lucas Sabalka, we took a different approach to the problem, and were able to prove:

Theorem 1.1. [SZ16] *Belk's space $\mathcal{C}F$ is a classifying space for F.*

One key step was to make use of a certain CAT(0) *cube complex* on which F acts, first studied by Stein and Farley.

It is an active project of mine [Zar16a] to produce a concrete classifying space for the *braided Thompson's group* V_{br} of Brin and Dehornoy [Bri07, Deh06]. *Thompson's group* V is defined similarly to F, except the piecewise linear bijections need not be continuous, and the braided Thompson's group V_{br} is a melding of V with the family of braid groups. (In a strand diagram for V_{br}, the strands may split, merge, *and* braid.) A candidate classifying space for V_{br} is the space $\mathcal{C}V_{\mathrm{br}}$ of configurations of points in the plane $\mathbb{C}$ such that two points may coincide, but no three points may be too close, *and* also no two points may too close except horizontally.

Question 1.2. *Is $\mathcal{C}V_{\mathrm{br}}$ is a classifying space for V_{br}?*

2. Finiteness properties of infinite groups

If a group G admits a classifying space whose n-skeleton is compact, we say G is of *type* F_n. Every group is of type F_0, a group is finitely generated if and only if it is of type F_1, and is finitely presented if and only if it is of type F_2. Hence these topological *finiteness properties* are natural extensions of these classical group theoretic notions. We also say *type* F_∞ to mean F_n for all n.

In a seminal 1987 paper [Bro87], Brown established a necessary and sufficient criterion for determining a group's finiteness properties based on its action on a CW-complex. Ten years later, Bestvina and Brady [BB97] developed a combinatorial version of Morse theory that has since proved to be an invaluable tool in applying Brown's criterion to certain groups. A number of my projects, which I will discuss now, have involved computing the finiteness properties of certain groups of interest, utilizing Brown's criterion and Bestvina–Brady Morse theory.

The *Brin–Thompson groups* sV ($s \in \mathbb{N}$) were introduced by Brin in a 2004 paper [Bri04], and were shown to be of type F_∞ for small values of s in a 2013 paper of Kochloukova, Martínez-Pérez and Nucinkis [KMPN13], using the action of sV on a certain topological space, but the techniques used there became unfeasible for large s. In joint work with Fluch, Marschler and Witzel, we used the action of sV on a smaller, more manageable space to prove:

Theorem 2.1. [FMWZ13] *The Brin–Thompson groups* sV *are of type* F_∞ *for all* $s \in \mathbb{N}$.

Using Brown's criterion and Bestvina–Brady Morse theory, applied to this manageable space, the problem reduced to understanding the topology of a certain family of finite complexes, which we did using powerful techniques of Quillen [Qui78].

We implemented a similar strategy for the aforementioned braided Thompson's group V_{br}, and its relative F_{br}, but the analogous complexes were not finite, and were much more difficult to analyze. These complexes, called *matching complexes of arcs on surfaces*, are a melding of the classical *matching complexes on graphs* and *arc complexes on surfaces*. Utilizing new, intricate techniques, partially building off suggestions of Andy Putman, we (the previous authors plus Bux) were able to pin down the topology of these complexes and ultimately proved:

Theorem 2.2. [BFM$^+$16] *The braided Thompson's groups* V_{br} *and* F_{br} *are of type* F_∞.

This result had been conjectured by Meier over a decade prior. Our approach to proving Theorems 2.1 and 2.2 has also inspired subsequent work of others, e.g., by Belk–Matucci, Martínez-Pérez–Matucci–Nucinkis, Nucinkis–St. John Green, and Thumann in understanding finiteness properties of "Thompson-like" groups.

Spurred by the techniques used in [BFM$^+$16], Witzel and I developed a framework, called *cloning systems*, to produce new families of Thompson-like groups $\mathscr{T}(G_*)$, out of interesting families of groups $(G_n)_{n\in\mathbb{N}}$ [WZ16b]. Existing examples come from the braid groups, symmetric groups, and others. Interesting new examples we explicitly discuss in [WZ16b] include *loop braid groups*, *mock symmetric groups*, and *upper triangular matrix groups*. Without getting into any details, "cloning" describes the interaction between the elements of the groups G_n of interest and a family of *splitting moves* coming from Thompson's groups. See Figure 3 for a flavor of "cloning" in the loop braid groups (the context for the picture can be found in [WZ16b]).

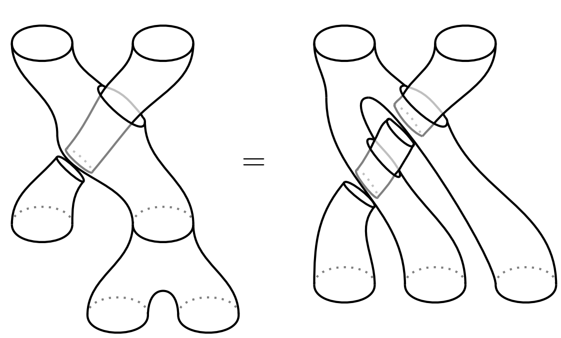

Figure 3. A visualization of "cloning" in the loop braid groups.

The finiteness properties of $\mathscr{T}(G_*)$ are often the limit of those for the G_n. This is unusual behavior for a limiting process, for example the direct limit usually destroys finiteness properties. We proved this for some examples in [WZ16b] using new techniques for proving finiteness properties, and we hope to further develop these tools in the future.

Question 2.3. *How often are the finiteness properties of $\mathscr{T}(G_*)$ the limit of those for the G_n?*

Our cloning systems framework was also used by a team of undergraduates, in an REU at Miami University, to produce potential counterexamples to the conjecture that V is a *universal co*$\mathcal{CF}$ *group* [BZFGM14]. In the summer of 2016 I also ran an independent study with my student Eidan Maimoni, constructing new, interesting examples of groups from cloning systems, e.g., a *ribbon braided Thompson group.*

Studying these cloning system groups continues to be an exciting new avenue of research. Besides the goals of building more examples and finding their finiteness properties, there also are some connections to the recent developments of Church, Ellenberg and Farb on *FI-modules* and *representation stability* [CEF15], and I would like to pin this down.

It turns out that Morse theory often does not apply when one is trying to prove a group *does not* have a certain finiteness property. In a recent paper, Belk and Forrest found a group T_B of homeomorphisms of the *Basilica Julia set*, which they proved is finitely generated and has a simple subgroup of finite index [BF15]. These properties make T_B a very interesting addition to the pantheon of infinite, virtually simple groups, and Belk and Forrest conjectured that T_B is *not* finitely presented. This being a negative property, the Morse theoretic approach was inconclusive. However, by developing new techniques for finiteness properties using CAT(0) cube complexes, Witzel and I confirmed this conjecture:

Theorem 2.4. [WZ16a] *The Basilica Thompson group T_B is not finitely presented.*

In the course of our work we were led to pose the following (vague) open question:

Question 2.5. *Do there exist simple Thompson-like groups that are finitely presented but not of type* F_∞*?*

This is a tantalizing question since finding new examples of infinite simple groups is always a desirable goal, and finiteness properties are a convenient way to tell whether an example is truly different than previously existing examples.

There are some groups for which it is already a very difficult problem to tell whether or not they are even finitely *generated*. A concrete example is $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$, the group of 2-by-2 matrices with Laurent polynomial entries and determinant 1. For decades, people have tried to prove that this is (or is not) finitely generated, but all the algebraic approaches so far have been inconclusive. It is known, thanks to a 1997 result of Krstić and McCool [KM97], that the group is *not* finitely presented. In a 2006 paper, Bux and Wortman [BW06] set up a geometric approach to the problem, and recovered the Krstić–McCool result. It would be very exciting to finish this problem, for example by developing a variation of Morse theory.

Question 2.6. *Is* $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$ *finitely generated?*

3. Topological properties at infinity

Knowing the finiteness properties of a group gives us an idea of how nice of a classifying space the group could have. Once a nice, concrete classifying space X is in hand, new questions arise. For example, one can ask about the *topological properties at infinity* of the universal cover $\widetilde{X}$.

These reveal many important features of the group and its subgroups. "Topological properties at infinity" is a vague term, with more than one meaning. I will focus on two aspects of topology at infinity, with respect to my research.

The first topological property "at infinity" is the notion of *homotopy groups at infinity* for a space Y. These arise by taking a limit, over all compact subspaces $K \subset Y$, of the usual homotopy groups π_k of the complement $Y \setminus K$. Hence, the homotopy groups at infinity only detect topological behavior in the space that *persists* outside of any compact subspace. If the universal cover of a classifying space for a group G has trivial homotopy groups at infinity, we say G itself has *vanishing homotopy groups at infinity*.

My research involving homotopy groups at infinity is as follows. In the 1970's, Geoghegan made four conjectures about Thompson's group F: (1) It is non-amenable, (2) It has no non-abelian free subgroups, (3) It is of type F_∞, and (4) It has vanishing homotopy groups at infinity. Properties (2), (3) and (4) were proved for F by Brown–Geoghegan [BG84] and Brin–Squier [BS85]. Property (1) for F remains infamously open, and has proven to be an incredibly difficult problem. Until recently, no group was known to satisfy all four of Geoghegan's properties. Roughly, properties (1) and (4) say the group is "not too small" and properties (2) and (3) say it is "not too big" so to satisfy all of them requires a delicate balancing act.

In a recent paper [LM16], Lodha and Moore introduced a remarkable group, which I call the *Lodha–Moore group* LM, that was the first tractable example of a finitely presented group satisfying properties (1) and (2) above. Lodha also proved that LM has property (3) [Lod14], bringing it "one step away" from being the first example of a group with all four of Geoghegan's properties. The only open question was property (4), which I was able to resolve in [Zar16b].

Theorem 3.1. [Zar16b] *The homotopy groups at infinity for LM vanish, i.e., LM satisfies property (4), and hence LM is the first known example of a group satisfying all four of Geoghegan's properties.*

The Lodha–Moore group was already very interesting as the first straightforward example of a finitely presented group with properties (1) and (2), and it is still more fascinating as the first known example of a group with all four of these properties.

Another notion of topological properties "at infinity" comes from the *Bieri–Neumann–Strebel–Renz invariants* of a group. Let G be a group of type F_n and let X be the compact n-skeleton of some classifying space for G. Any non-trivial *character* of G, i.e., a homomorphism χ from G to $\mathbb{R}$, induces a measurement $h_\chi \colon \widetilde{X} \to \mathbb{R}$ on the universal cover $\widetilde{X}$. This in turn specifies a *direction toward infinity* in the space, namely, if we travel through $\widetilde{X}$ and h_χ gets larger and larger, we are going "toward infinity" relative χ. Now, if the space is *essentially* $(n-1)$*-connected in this direction*, we say χ is in the *Bieri–Neumann–Strebel–Renz invariant* $\Sigma^n(G)$.

This is admittedly quite complicated, but the intuition is that the $\Sigma^m(G)$, for $m \le n$, are a catalog of which directions toward infinity yield vanishing homotopy groups up to dimension $m-1$. The invariant $\Sigma^1(G)$ was introduced by Bieri, Neumann and Strebel [BNS87] in 1987 and the $\Sigma^m(G)$ for $m \ge 2$ by Bieri and Renz [BR88] in 1988. The BNSR-invariants are nested:

$$\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots \supseteq \Sigma^\infty(G),$$

and are all subsets of the *character sphere* $S(G)$ of G. The powerful main application of the invariants $\Sigma^m(G)$ is that they determine the finiteness properties of *every* subgroup of G containing the commutator subgroup $[G,G]$. For example, $[G,G]$ itself is finitely generated if and only *every* character of G lies in $\Sigma^1(G)$.

Because they reveal so much information, it is extremely useful to know the BNSR-invariants of a group. However, in general they are exceedingly difficult to compute. Until recently, the only robust family of groups for which the problem is relevant, difficult and totally solved is *right-angled Artin groups*. This was proved independently by Meier–Meinert–VanWyk and Bux-Gonzalez in the late 1990's. Here a *right-angled Artin group* is a group admitting a finite presentation in which the defining relations say that certain pairs of generators commute, for example free groups F_n and free abelian groups $\mathbb{Z}^n$ are right-angled Artin groups.

I recently added to the list of groups with known BNSR-invariants. Thompson's group F fits into a family of groups $F_{n,\infty}$, of which F is $F_{2,\infty}$. One can define $F_{n,\infty}$ by mimicking the definition of F in Section 1, as bijections of $[0,1]$, but with n in place of 2 everywhere. In a 2010 paper, Bieri, Geoghegan and Kochloukova [BGK10] were able to compute all the BNSR-invariants $\Sigma^m(F)$ of F. This led to the natural problem of computing $\Sigma^m(F_{n,\infty})$ for all m and n. For $n > 2$ however, the techniques in [BGK10] could not yield a full computation. Kochloukova [Koc12] was able to compute $\Sigma^2(F_{n,\infty})$ for all n, using a different approach, but this still could not handle the case when m and n are *both* larger than 2. Using intricate Morse theoretic tools, employed on certain CAT(0) cube complexes (developed by Stein and Farley), I was able to prove:

Theorem 3.2. [Zar16c] *For all $n, m \geq 2$ we have $\Sigma^m(F_{n,\infty}) = \Sigma^2(F_{n,\infty})$.*

Since Kochloukova computed $\Sigma^2(F_{n,\infty})$, this finishes the problem. My approach was inspired by joint work with Witzel [WZ15b], where we redid Bieri, Geoghegan and Kochloukova's computation of $\Sigma^m(F)$ using Morse theory and the Stein–Farley CAT(0) cube complexes.

One reason this problem was so difficult is that the BNSR-invariants of $F_{n,\infty}$ lie in an $(n-1)$-sphere, so for large n the sphere becomes so high-dimensional that BNSR-invariants encode a massive amount of information. A similar problem occurs for the BNSR-invariants of the *pure braid groups* P_n. These are the subgroups $P_n \leq B_n$ consisting of braids in which each strand ends up in the same position at the bottom it was in at the top. While the BNSR-invariants of B_n lie in a 0-sphere for all n, and are easy to compute (they all equal the whole 0-sphere), the BNSR-invariants of P_n lie in an $(\binom{n}{2} - 1)$-sphere. This dimension grows even faster than the $(n-1)$-spheres for the $F_{n,\infty}$, and the problem is correspondingly even harder. It is not solved yet, but Koban, McCammond and Meier computed $\Sigma^1(P_n)$ for all n [KMM15], and I proved:

Theorem 3.3. [Zar16d] *For all $3 \leq m \leq n$ the inclusion $\Sigma^{m-2}(P_n) \subseteq \Sigma^{m-3}(P_n)$ is proper, and for all $n < m$ it is an equality.*

A pleasant consequence that arose in [Zar16d] is the following: consider the subgroup H of pure braids in which the total number of times the first and second strands wind around each other equals the total number of times the third and fourth strands wind around each other. Then H is finitely generated (type F_1) but not finitely presented (not type F_2). I found similar subgroups of type F_{m-3} but not F_{m-2}, for all $3 \leq m \leq n$. It is a testament to the power of all these topological techniques that we can say H is finitely generated and not finitely presented, without ever writing down a presentation, or even a generating set. In the future I hope to get a complete computation of all the $\Sigma^m(P_n)$.

Another family of groups for while I have some results on their BNSR-invariants is the aforementioned (pure) loop braid groups, also called groups of *(pure) symmetric* free group automorphisms. A free group automorphism is called symmetric if it sends each basis element to a conjugate of a basis element, and pure symmetric if it sends each basis element to a conjugate of itself. The groups $\Sigma\mathrm{Aut}_n$ and $P\Sigma\mathrm{Aut}_n$ of such automorphisms appear in many disparate contexts and have interesting, important properties. The BNS-invariant $\Sigma^1(P\Sigma\mathrm{Aut}_n)$ was found by Orlandi-Korner in 2000 [OK00], and I obtained some partial results in [Zar16e] for $\Sigma^m(P\Sigma\mathrm{Aut}_n)$ ($m \geq 2$). The results for $P\Sigma\mathrm{Aut}_n$ actually allowed me to fully compute the BNSR-invariants of $\Sigma\mathrm{Aut}_n$:

Theorem 3.4. [Zar16e] *For $n \geq 2$, we have $\Sigma^{n-2}(\Sigma Aut_n) = S(\Sigma Aut_n) = S^0$ and $\Sigma^{n-1}(\Sigma Aut_n) = \emptyset$. In particular the commutator subgroup $\Sigma Aut_n'$ is of type F_{n-2} but not F_{n-1}.*

For example $\Sigma\mathrm{Aut}_n'$ is finitely generated if and only if $n \geq 3$, and finitely presentable if and only if $n \geq 4$, which was already a new result. This also provides the first natural examples for $m \geq 2$ of groups G of type F_∞ such that $\Sigma^{m-1}(G) = S(G)$ but $\Sigma^m(G) = \emptyset$, and of groups of type F_∞ whose commutator subgroups have arbitrary finiteness properties.

4. Homological stability

As mentioned in Section 1, one of the reasons classifying spaces are so important is that they reveal the homology of a group. It is often difficult to compute a group's homology, or even to deduce basic properties of it, such as whether it is trivial or not. The translation to the topological world can make difficult questions more tractable. For example, once we are in the topological realm, tools like Morse theory become possible.

An example of this is *homological stability* for a family of groups. A family of groups G_n with homomorphisms $G_n \to G_{n+1}$ is called homologically stable in n provided that, for all i and all $n >> i$, the map $G_n \to G_{n+1}$ induces an isomorphism in the ith homology,

$$H_i(G_n) \stackrel{\cong}{\to} H_i(G_{n+1}).$$

Homological stability is a desirable property to have, since for each i it shrinks an infinite list of objects to a finite list. It can be difficult to say anything about stability of group homology, but if one has a family of classifying spaces in hand, for each G_n, then this translates to the more concrete question of stability for the homology of topological spaces.

Homological stability is an active and fruitful area of research. Many classical families of groups are homologically stable, for example symmetric groups, braid groups, mapping class groups, and, the one I will focus on now, $\mathrm{Aut}(F_n)$.

Theorem 4.1 (Hatcher–Vogtmann)**.** [HV98] *The groups $\mathrm{Aut}(F_n)$ are homologically stable.*

In a 1998 paper, Hatcher and Vogtmann [HV98] established homological stability for the groups $\mathrm{Aut}(F_n)$ of *automorphisms of the free groups F_n*. The groups $\mathrm{Aut}(F_n)$, and the groups of *outer automorphisms* $\mathrm{Out}(F_n)$, are fundamental objects of study in modern geometric group theory. A milestone in the study of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ was the introduction, in a 1986 paper of Culler and Vogtmann [CV86], of the so called *Culler–Vogtmann Outer space* and its simplicial *spine*. This spine is a contractible simplicial complex on which $\mathrm{Out}(F_n)$ acts with finite stabilizers and compact quotient. There is a similar space, called *Auter space*, for $\mathrm{Aut}(F_n)$. Hatcher and Vogtmann used Auter space to deduce homological stability for $\mathrm{Aut}(F_n)$, by translating the problem into a topological one.

The key result in Hatcher and Vogtmann's paper was the *Degree Theorem*, which states that certain subspaces of Auter space have vanishing homotopy groups up to some dimension. The proof of the Degree Theorem in [HV98] was done by globally deforming homotopy spheres, and is quite intricate, involving many steps. In joint work with McEwen [MZ14] we reproved the Degree Theorem using Bestvina–Brady Morse theory, which reduced the global problem to a simpler local one. Our approach is also more readily generalizable to other contexts, for instance to questions of stability for partially symmetric automorphism groups.

An automorphism of F_{m+n} is called *partially symmetric* if it sends the first m generators of F_{m+n} to conjugates of each other, or their inverses. If it sends these generators to conjugates of themselves, call it *pure partially symmetric*. Let $\Sigma\mathrm{Aut}_n^m$ denote the subgroup of $\mathrm{Aut}(F_{m+n})$ consisting of partially symmetric automorphisms, and $\mathrm{P}\Sigma\mathrm{Aut}_n^m$ the pure partially symmetric automorphisms. Now there are two parameters, m and n, in which the groups could be homologically stable. The question of stability for $\Sigma\mathrm{Aut}_n^m$ and/or $\mathrm{P}\Sigma\mathrm{Aut}_n^m$ becomes even more interesting knowing that, when $m = 0$, $\Sigma\mathrm{Aut}_n^0 = \mathrm{Aut}(F_n)$ is stable in n, but when $n = 0$, $\mathrm{P}\Sigma\mathrm{Aut}_0^m$ is *not* stable in m.

Expounding on the Morse theoretic techniques from [MZ14], I was able to prove:

Theorem 4.2. [Zar14] *The rational homology of* $\Sigma\mathrm{Aut}_n^m$ *is stable in both* m *and* n*, and the rational homology of* $\mathrm{P}\Sigma\mathrm{Aut}_n^m$ *is stable in* n*.*

In order to prove these stability results, I used a version of Auter space suited to the partially symmetric automorphism groups. The Morse theoretic approach from [MZ14] could have additional future applications, and in general the partially symmetric automorphism groups $\Sigma\mathrm{Aut}_n^m$ are an important family of groups, interpolating between the often mysterious full groups of automorphisms (when $m = 0$) and the more well understood groups of symmetric automorphisms (when $n = 0$).

5. Groups acting on buildings

Another topic that plays a prominent role in my research is building theory. Buildings were originally developed by Jacques Tits as a unified tool to analyze algebraic groups over arbitrary fields, and since then many important applications for building theory have arisen.

A *building* is a simplicial complex, covered by subcomplexes called *apartments*, such that any two simplices share a common apartment, any two apartments are isomorphic via an isomorphism fixing their intersection, and each apartment is a *Coxeter complex*. Coxeter complexes are, topologically, spaces like spheres or planes, with a geometry informed by reflections along hyperplanes. The group generated by these reflections is the *Coxeter group* W of the building.

Given a group G acting on a building Δ, there are two notions of transitivity that play an important role. If G is transitive on maximal simplices, called *chambers*, and for each chamber C the stabilizer of C in G is transitive on apartments containing C, we call the action *strongly transitive*. If instead it is only transitive on $\{D \mid \delta(C, D) = w\}$ for all $w \in W$, we call the action *Weyl transitive*. Here δ is the *Weyl distance function*, which assigns an element of the Coxeter group W to each pair of chambers in a certain way. Strong transitivity implies Weyl transitivity, but it was not known until recently whether the converse was true. Abramenko and Brown found examples of Weyl transitive actions on *trees* that are not strongly transitive [AB07], but it was unclear whether this behavior was specific to these 1-dimensional examples. My PhD thesis work produced examples of such actions for buildings of arbitrary dimension, and with arbitrary affine Coxeter group W.

Theorem 5.1. [AZ11, Zar15] *For any affine* W*, there are examples of groups acting Weyl transitively on buildings with Coxeter group* W*, such that the action is not strongly transitive.*

Any time one has a group acting on a building, the huge amount of symmetry in the building makes this a very advantageous situation for understanding the group. An example of this is my work, joint with Witzel, on the *Burau representation* of the braid group B_4 [WZ15a]. The Burau representation $\rho_4 \colon B_4 \to \mathrm{SL}_3(\mathbb{Z}[t, t^{-1}])$ is a certain way of viewing 4-strand braids as 3-by-3 matrices, with Laurent polynomial entries. Whether or not ρ_4 is injective has been a major open problem for decades. The corresponding representations ρ_n for $n > 4$ are not injective, and for $n < 4$ they are, so $n = 4$ is the only question. This question also has implications for the famous problem of whether the Jones polynomial detects the unknot. In the 1970's Birman reduced the injectivity question to proving that a pair of explicitly given matrices $f, k \in \mathrm{SL}_3(\mathbb{Z}[t, t^{-1}])$ generate a copy of the free group F_2.

In [WZ15a], we used the action of $\mathrm{SL}_3(\mathbb{Z}[t, t^{-1}])$ on a certain affine building, and the well known Ping-Pong Lemma, to prove that:

Theorem 5.2. [WZ15a] *For any $m, n \geq 3$, we have $\langle f^m, k^n \rangle \cong F_2$.*

This does not immediately say anything about whether or not $\langle f, k \rangle \cong F_2$, or whether ρ_4 is faithful, but it is possible that our techniques could be improved in the future to get the Ping-Pong argument to work for f and k themselves.

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