MATH 3110: HOMEWORK 7

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

Problem 0 (don't submit any of this problem for a grade)**.** This problem consists of some "definition-chasing" exercises that should help you feel more comfortable with the new concepts.

(a) Suppose that $f\colon A \to \mathbf{R}$ is a function and $B \subseteq A$. The *restriction* of f to B, denoted $f \upharpoonright B$, is the function with domain B given by, for $x \in B$,

$$(f \upharpoonright B)(x) = f(x).$$

(It's just "the same function" with a restricted domain.) Prove that if f is continuous, then so is $f \upharpoonright B$.

(b) Verify that, in the definition of *continuity*, if a δ "response" meets an ε "challenge," then any smaller δ' also meets the same ε challenge. Verify also that δ meets any challenge $\varepsilon' > \varepsilon$.

(c) Let $f\colon A \to \mathbf{R}$ be a function. Verify that the following definitions of continuity are equivalent.

(i) for all $c \in A$, for all $\varepsilon > 0$, there is $\delta > 0$ such that
for all $x \in A$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

(ii) for all $c \in A$, for all $\varepsilon \in (0, 1)$, there is $\delta > 0$ such that
for all $x \in A$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

(iii) for all $c \in A$, for all $\varepsilon > 0$, there is $\delta > 0$ such that
for all $x \in A \cap B_\delta(c)$, we have $f(x) \in B_\varepsilon(f(c))$.

(iv) for all $c \in A$, for all $\varepsilon > 0$, there is $\delta > 0$ such that
$f[B_\delta(c) \cap A] \subseteq B_\varepsilon(f(c))$.

(v) for all $c \in A$, for all $\varepsilon > 0$, there is $\delta > 0$ such that
for all $x, y \in A \cap B_\delta(c)$, we have $|f(x) - f(y)| < \varepsilon$.

(d) Show that for a function $f\colon \mathbf{R} \to \mathbf{R}$, the following are equivalent to continuity.

(vi) for all $c \in \mathbf{R}$, for all $\varepsilon > 0$, there is $\delta > 0$ such that
for all x, $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

(vii) for all open sets $U \subseteq \mathbf{R}$, the preimage $f^{-1}[U]$ is open.

(viii) for all sets $B \subseteq \mathbf{R}$, we have $f[\overline{B}] \subseteq \overline{f[B]}$

Date: Due Wednesday, 20 March 2019.

Problem 1. Prove that if $f: A \to \mathbf{R}$ and $g: B \to \mathbf{R}$ are continuous, then so are the functions

(a) $|f|$,
(b) $\max(f, g)$, and
(c) $\min(f, g)$.

(One might deduce (b) & (c) from (a)...)

Problem 2. In class, we proved that if $f: A \to \mathbf{R}$ and $g: B \to \mathbf{R}$ are each continuous at $c \in A \cap B$, then the pointwise sum $f + g: A \cap B \to \mathbf{R}$ is continuous at c. Similar facts are also true for products and quotients of functions. Prove them. (*Hint*: Use the sequential characterization of continuity.)

Problem 3. Give a careful proof that all polynomials and rational functions are continuous, using what we proved in lecture. That is, assuming that constant functions and the function $f(x) = x$ are continuous and that sums, products, and quotients of continuous functions are continuous, prove that all polynomials and all rational functions are continuous.

Problem 4. Give an alternative proof of the Intermediate Value Theorem using the Nested Intervals Theorem.

(*Hint*: Google "bisection method." Also, there are hints in the textbook.)

Problem 5. Prove that if $f: [a, b] \to [a, b]$ is continuous, then there is $x \in [a, b]$ satisfying the equation $f(x) = x$.

Problem 6. For each of the following sets A, construct a continuous function $f: A \to \mathbf{R}$ that is unbounded on A. Explain why this does not contradict the Boundedness Theorem.

(a) $A = \mathbf{Z}$,
(b) $A = (0, 1)$,
(c) $A = [1, 2] \cap \mathbf{Q}$.

Problem 7. Prove that if $f: [a, b] \to \mathbf{R}$ is a one-to-one continuous function, then its inverse f^{-1} is a continuous function.

(*Hint*: A nice special case to start with is the square-root function. See if you can prove that that function is continuous. If you get stuck, it's done in the textbook.)