MATH 3110: HOMEWORK 9

You will be graded on both the accuracy and the clarity of your solutions. One purpose of the homework is to give you an opportunity to practice your proof-writing skills.

You are welcome — encouraged, even! — to collaborate on homework, but you should not copy solutions from any source, nor should you submit anything that you don't understand.

Problem 0 (don't submit any of this for a grade)**.**

(a) Review old homework problems and make sure you know how to do them. (But seriously.)
(b) Make sure you know how to deduce the standard corollaries to the Mean Value Theorem: a differentiable function is constant iff its derivative is 0, is increasing iff its derivative is never negative, etc.
(c) Prove that if $\lim_{x \to c} f(x) > 0$, then there is a neighborhood of c on which f is positive.

And here are some good practice problems from the textbook:

(d) 4.4.7, 4.4.13.
(e) 4.2.7–4.2.11.
(f) 5.3.2–5.3.5 and 5.3.7–5.3.8.

Problem 1. Prove that the function $x \mapsto \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Problem 2 (Constructing familiar functions II)**.** Review Homework 8, Problem 3. Fix a real number $b > 1$.

(⋆) (Don't turn anything in for this one.) What do you think 2^π should mean?
(a) Prove that if r is a rational number then

$$b^r = \sup\left\{b^t : t \in \mathbf{Q},\ t < r\right\}.$$

It is therefore consistent with our previous definition to define, for any $x \in \mathbf{R}$,

$$b^x = \sup\left\{b^t : t \in \mathbf{Q},\ t < x\right\}.$$

(b) Prove that for all $x, y \in \mathbf{R}$ we have $b^{x+y} = b^x \cdot b^y$.
(c) Prove that the function $\exp_b \colon \mathbf{R} \to \mathbf{R}$ defined by $\exp_b(x) = b^x$ is continuous and strictly increasing.
(d) Prove that the range of $\exp_b$ is the set $\{y \in \mathbf{R} : y > 0\}$.

(*Hint*: You might find it useful to prove that $\lim_{n \to \infty} b^{1/n} = 1$. For that, see Homework 2 Problem 3.)

Date: Due Monday (!), 15 April 2019.

Problem 3 (Compare with Homework 7, Problem 7)**.** Suppose that $f: [a,b] \to \mathbf{R}$ is a one-to-one differentiable function whose derivative is never 0: for all $x \in [a,b]$, $f'(x) \neq 0$. Prove that its inverse f^{-1} is differentiable with derivative given by

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Use this to produce the familiar formula for the derivative of the square-root function.

Problem 4 (essentially 4.3.11 from the textbook)**.** Suppose that $f: \mathbf{R} \to \mathbf{R}$ is a function, and assume that there is a constant $c \in (0,1)$ such that, for all $x, y \in \mathbf{R}$,

$$|f(x) - f(y)| \leq c\,|x - y|. \qquad (\star)$$

(a) Explain (in words) what our assumption $(\star)$ means geometrically.
(b) Prove that f is continuous (on $\mathbf{R}$).
(c) Show that if y_1 is any real number, then the sequence $(f^n(y_1))_{n\in\mathbf{N}}$ is Cauchy and therefore converges. ("f^n" means n applications of f, so $f^0(y_1) = y_1$ and $f^{n+1}(y_1) = f(f^n(y_1))$.)
(d) Let y_∞ be the limit of the sequence from the previous part. Prove that y_∞ is a fixed point of f (meaning $f(y_\infty) = y_\infty$) and that it is the *unique* fixed point of f.
(e) Conclude that if z_1 is any real number, then the sequence $(f^n(z_1))_{n\in\mathbf{N}}$ converges to the same y_∞ from the previous parts.
(f) ~~Assuming standard facts about the sine and cosine functions (which we haven't yet proved!), what does this problem tell you about sine and cosine?~~

Problem 5.

(a) Suppose that $g: [0,5] \to \mathbf{R}$ is differentiable, $g(0) = 0$, and g has bounded derivative on $[0,5]$: say $|g'(x)| \leq M$. Show for all $x \in [0,5]$ that $|g(x)| \leq Mx$.
(b) Suppose that $h: [0,5] \to \mathbf{R}$ is twice-differentiable (meaning that it's differentiable, and its derivative is differentiable), that $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$. Show for all $x \in [0,5]$ that $|h(x)| \leq Mx^2/2$.
(c) Can you give a geometric interpretation of the previous two parts?

Problem 6. For this problem, assume that $\mathrm{C}: \mathbf{R} \to \mathbf{R}$ is function with the following properties:

(i) C is differentiable;
(ii) for all $x \in \mathbf{R}$, $\mathrm{C}(x) \in [-1,1]$;
(iii) for all integers $n \in \mathbf{Z}$, $\mathrm{C}(2n) = 1$ and $\mathrm{C}(2n+1) = -1$ and $\mathrm{C}(n + \frac{1}{2}) = 0$.

(We might later prove that $x \mapsto \cos(\pi x)$ has these properties, but we haven't yet defined sine or cosine or π!!!)

(a) Prove that the function $f: \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} \mathrm{C}(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has the intermediate value property but is not continuous at $x = 0$.

(b) Prove that the function given by

$$g(x) = \begin{cases} x \cdot C(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$.

(*Hint*: Look at the animation on the course webpage.)

(c) Prove that the function given by

$$h(x) = \begin{cases} x^2\, C(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable everywhere (in particular at $x = 0$), but its derivative is discontinuous at $x = 0$.

(*Hint*: In the first part, you will show that $h'(0) = 0$. Use the Mean Value Theorem and item (iii) to find a sequence converging to 0 at which h' converges to 2.)