Contemporary Mathematics

A beginner's guide to forcing

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Dedicated to Joseph Gallian on his 65th birthday

1. Introduction

In 1963, Paul Cohen stunned the mathematical world with his new technique of *forcing,* which allowed him to solve several outstanding problems in set theory at a single stroke. Perhaps most notably, he proved the independence of the continuum hypothesis (CH) from the Zermelo-Fraenkel-Choice (ZFC) axioms of set theory. The impact of Cohen's ideas on the practice of set theory, as well as on the philosophy of mathematics, has been incalculable.

Curiously, though, despite the importance of Cohen's work and the passage of nearly fifty years, forcing remains totally mysterious to the vast majority of mathematicians, even those who know a little mathematical logic. As an illustration, let us note that Monastyrsky's outstanding book [**11**] gives highly informative and insightful expositions of the work of almost every Fields Medalist—but says almost nothing about forcing. Although there exist numerous textbooks with mathematically correct and complete proofs of the basic theorems of forcing, the subject remains notoriously difficult for beginners to learn.

All mathematicians are familiar with the concept of an *open research problem.* I propose the less familiar concept of an *open exposition problem.* Solving an open exposition problem means explaining a mathematical subject in a way that renders it totally perspicuous. Every step should be motivated and clear; ideally, students should feel that they could have arrived at the results themselves. The proofs should be "natural" in Donald Newman's sense [**13**]:

> This term ... is introduced to mean not having any ad hoc constructions or *brilliancies.* A "natural" proof, then, is one which proves itself, one available to the "common mathematician in the streets."

I believe that it is an open exposition problem to explain forcing. Current treatments allow readers to verify the truth of the basic theorems, and to progress fairly rapidly to the point where they can *use* forcing to prove their own independence results (see [**2**] for a particularly nice explanation of how to use forcing as a

2000 *Mathematics Subject Classification.* Primary 03E35; secondary 03E40, 00-01.

black box to turn independence questions into concrete combinatorial problems). However, in all treatments that I know of, one is left feeling that only a genius with fantastic intuition or technical virtuosity could have found the road to the final result.

This paper does not solve this open exposition problem, but I believe it is a step in the right direction. My goal is to give a rapid overview of the subject, emphasizing the broad outlines and the intuitive motivation while omitting most of the proofs. The reader will not, of course, master forcing by reading this paper in isolation without consulting standard textbooks for the omitted details, but my hope is to provide a map of the forest so that the beginner will not get lost while forging through the trees. Currently, no such bird's-eye overview seems to be available in the published literature; I hope to fill this gap. I also hope that this paper will inspire others to continue the job of making forcing totally transparent.

2. Executive summary

The negation of CH says that there is a cardinal number, $\aleph_1$, between the cardinal numbers $\aleph_0$ and $2^{\aleph_0}$. One might therefore try to build a structure that satisfies the negation of CH by starting with something that *does* satisfy CH (Gödel had in fact constructed such structures) and "inserting" some sets that are missing.

The fundamental theorem of forcing is that, under very general conditions, one can indeed start with a mathematical structure M that satisfies the ZFC axioms, and enlarge it by adjoining a new element U to obtain a new structure $M[U]$ that also satisfies ZFC. Conceptually, this process is analogous to the process of adjoining a new element X to, say, a given ring R to obtain a larger ring $R[X]$. However, the construction of $M[U]$ is a lot more complicated because the axioms of ZFC are more complicated than the axioms for a ring. Cohen's idea was to build the new element U one step at a time, tracking what new properties of $M[U]$ would be "forced" to hold at each step, so that one could control the properties of $M[U]$—in particular, making it satisfy the negation of CH as well as the axioms of ZFC.

The rest of this paper fleshes out the above construction in more detail.

3. Models of ZFC

As mentioned above, Cohen proved the independence of CH from ZFC; more precisely, he proved that if ZFC is consistent, then CH is not a logical consequence of the ZFC axioms. Gödel had already proved that if ZFC is consistent, then $\neg$CH, the negation of CH, is not a logical consequence of ZFC, using his concept of "constructible sets." (Note that the hypothesis that ZFC is consistent cannot be dropped, because if ZFC is inconsistent then *everything* is a logical consequence of ZFC!)

Just how does one go about proving that CH is not a logical consequence of ZFC? At a very high level, the structure of the proof is what you would expect: One writes down a very precise statement of the ZFC axioms and of $\neg$CH, and then one constructs a mathematical structure that satisfies both ZFC and $\neg$CH. This structure is said to be a *model* of the axioms. Although the term "model" is not often seen in mathematics outside of formal logic, it is actually a familiar concept. For example, in group theory, a "model of the group-theoretic axioms" is just a group, i.e., a set G with a binary operation $*$ satisfying axioms such as: "There exists an element e in G such that $x * e = e * x = x$ for all x in G," and so forth.

Analogously, we could invent a term—say, *universe*—to mean "a structure that is a model of ZFC." Then we could begin our study of ZFC with definition such as, "A *universe* is a set M together with a binary relation R satisfying…" followed by a long list of axioms such as the *axiom of extensionality:*

> If x and y are distinct elements of M then either there exists z in M such that zRx but not zRy, or there exists z in M such that zRy but not zRx.

Another axiom of ZFC is the *powerset axiom:*

> For every x in M, there exists y in M with the following property: For every z in M, zRy if and only if $z \subseteq x$.

(Here the expression "$z \subseteq x$" is an abbreviation for "every w in M satisfying wRz also satisfies wRx.") There are other axioms, which can be found in any set theory textbook, but the general idea should be clear from these two examples. Note that the binary relation is usually denoted by the symbol $\in$ since the axioms are inspired by the set membership relation. However, we have deliberately chosen the unfamiliar symbol R to ensure that the reader will not misinterpret the axiom by accidentally reading $\in$ as "is a member of."

As an aside, we should mention that it is not standard to use the term *universe* to mean "model of ZFC." For some reason set theorists tend to give a snappy name like "ZFC" to a *list of axioms,* and then use the term "model of ZFC" to refer to the *structures that satisfy the axioms*, whereas in the rest of mathematics it is the other way around: one gives a snappy name like "group" to the structure, and then uses the term "axioms for a group" to refer to the axioms. Apart from this terminological point, though, the formal setup here is entirely analogous to that of group theory. For example, in group theory, the statement S that "$x * y = y * x$ for all x and y" is not a logical consequence of the axioms of group theory, because there exists a mathematical structure—namely a non-abelian group—that satisfies the group axioms as well as the negation of S.

On the other hand, the definition of a model of ZFC has some curious features, so a few additional remarks are in order.

3.1. Apparent circularity. One common confusion about models of ZFC stems from a tacit expectation that some people have, namely that we are supposed to suspend all our preconceptions about sets when beginning the study of ZFC. For example, it may have been a surprise to some readers to see that a universe is defined to be a *set* together with…. Wait a minute—what is a set? Isn't it circular to define sets in terms of sets?

In fact, we are not defining sets in terms of sets, but *universes* in terms of sets. Once we see that all we are doing is studying a subject called "universe theory" (rather than "set theory"), the apparent circularity disappears.

The reader may still be bothered by the lingering feeling that the point of introducing ZFC is to "make set theory rigorous" or to examine the foundations of mathematics. While it is true that ZFC can be used as a tool for such philosophical investigations, we do not do so in this paper. Instead, we take for granted that ordinary mathematical reasoning—including reasoning about sets—is perfectly valid and does not suddenly become invalid when the object of study is ZFC. That is, we approach the study of ZFC and its models in the same way that one approaches the study of any other mathematical subject. This is the best way to grasp the mathematical content; after this is achieved, one can then try to apply the technical results to philosophical questions if one is so inclined.

Note that in accordance with our attitude that ordinary mathematical reasoning is perfectly valid, we will freely employ reasoning about infinite sets of the kind that is routinely used in mathematics. We reassure readers who harbor philosophical doubts about the validity of infinitary set-theoretic reasoning that Cohen's proof can be turned into a purely finitistic one. We will not delve into such metamathematical niceties here, but see for example the beginning of Chapter VII of Kunen's book [**10**].

3.2. Existence of examples. A course in group theory typically begins with many examples of groups. One then verifies that the examples satisfy all the axioms of group theory. Here we encounter an awkward feature of models of ZFC, which is that exhibiting explicit models of ZFC is difficult. For example, there are no finite models of ZFC. Worse, by a result known as the *completeness theorem,* the statement that ZFC has *any models at all* is equivalent to the statement that ZFC is consistent, which is an assumption that is at least mildly controversial. So how can we even get off the ground?

Fortunately, these difficulties are not as severe as they might seem at first. For example, one entity that is almost a model of ZFC is V, the class of all sets. If we take $M = V$ and we take R to mean "is a member of," then we see that the axiom of extensionality simply says that two sets are equal if and only if they contain the same elements—a manifestly true statement. The rest of the axioms of ZFC are similarly self-evident when $M = V$.[1] The catch is that a model of ZFC has to be a *set,* and V, being "too large" to be a set (Cantor's paradox), is a proper class and therefore, strictly speaking, is disqualified from being a model of ZFC. However, it is close enough to being a model of ZFC to be intuitively helpful.

As for possible controversy over whether ZFC is consistent, we can sidestep the issue simply by treating the consistency of ZFC like any other unproved statement, such as the Riemann hypothesis. That is, we can assume it freely as long as we remember to preface all our theorems with a conditional clause.[2] So from now on we shall assume that ZFC is consistent, and therefore that models of ZFC exist.

3.3. "Standard" models. Even granting the consistency of ZFC, it is not easy to produce models. One can extract an example from the proof of the completeness theorem, but this example is unnatural and is not of much use for tackling CH. Instead of continuing the search for explicit examples, we shall turn our attention to important properties of models of ZFC.

One important insight of Cohen's was that it is useful to consider what he called *standard* models of ZFC. A model M of ZFC is standard if the elements of M are *well-founded sets* and if the relation R is ordinary set membership. Well-founded sets are sets that are built up inductively from the empty set, using operations such as taking unions, subsets, powersets, etc. Thus the empty set $\{\}$ is well-founded, as are $\{\{\}\}$ and the infinite set $\{\{\}, \{\{\}\}, \{\{\{\}\}\}, \ldots\}$. They are called "well-founded" because the nature of their inductive construction precludes any well-founded set from being a member of itself. We emphasize that if M is standard, then the

[1] Except, perhaps, the axiom of regularity, but this is a technical quibble that we shall ignore.

[2] In fact, we already did this when we said that the precise statement of Cohen's result is that *if ZFC is consistent* then CH is not a logical consequence of ZFC.

elements of M are not amorphous "atoms," as some of us envisage the elements of an abstract group to be, but are *sets.* Moreover, well-founded sets are *not* themselves built up from "atoms"; it's "sets all the way down."

While it is fairly clear that if standard models of ZFC exist, then they form a natural class of examples, it is not at all clear that any standard models exist at all, even if ZFC is consistent.[3] (The class of all well-founded sets is a proper class and not a set and hence is disqualified.) Moreover, even if standard models exist, one might think that constructing a model of ZFC satisfying ¬CH might require considering "exotic" models in which the binary relation R bears very little resemblance to ordinary set membership. Cohen himself admits on page 108 of [**6**] that a minor leap of faith is involved here:

> Since the negation of CH or AC may appear to be somewhat unnatural one might think it hopeless to look for standard models. However, we make a firm decision at the point to consider only standard models. Although this may seem like a very severe limitation in our approach it will turn out that this very limitation will guide us in suggesting possibilities.

Another property that a model of ZFC can have is *transitivity.* A standard model M of ZFC is *transitive* if every member of an element of M is also an element of M. (The term *transitive* is used because we can write the condition in the suggestive form "$x \in y$ and $y \in M$ implies $x \in M$.") This is a natural condition to impose if we think of M as a universe consisting of "all that there is"; in such a universe, sets "should" be sets of things that already exist in the universe. Cohen's remark about standard models applies equally to transitive models.[4]

Our focus will be primarily on standard transitive models. Of course, this choice of focus is made with the benefit of hindsight, but even without the benefit of hindsight, it makes sense to study models with natural properties before studying exotic models. When M is a standard transitive model, we will often use the symbol $\in$ for the relation R, because in this case R *is* in fact set membership.

4. Powersets and absoluteness

At some point in their education, most mathematicians learn that all familiar mathematical objects can be defined in terms of sets. For example, one can define the number 0 to be the empty set $\{\}$, the number 1 to be the set $\{0\}$, and in general the number n to be the so-called *von Neumann ordinal* $\{0, 1, \ldots, n-1\}$. The set $\mathbb{N}$ of all natural numbers may be defined to be $\aleph_0$, the set of all von Neumann ordinals.[5] Note that with these definitions, the membership relation on $\aleph_0$ corresponds to the usual ordering on the natural numbers (this is why n is defined as $\{0, 1, \ldots, n-1\}$ rather than as $\{n-1\}$). The ordered pair (x, y) may be defined à la Kuratowski as the set $\{\{x\}, \{x, y\}\}$. Functions, relations, bijections, maps, etc., can be defined as certain sets of ordered pairs. More interesting mathematical structures can be defined as ordered pairs (X, S) where X is an underlying set and S is the structure

[3] It turns out that the existence of a standard model of ZFC is indeed a stronger assumption than the consistency of ZFC, but we will ignore this nicety.

[4] We remark in passing that the *Mostowski collapsing theorem* implies that if there exist *any* standard models of ZFC, then there exist standard transitive models.

[5] We elide the distinction between the cardinality $\aleph_0$ of $\mathbb{N}$ and the order type ω of $\mathbb{N}$.

on X. With this understanding, the class V of all sets may be thought of as being the entire mathematical universe.

Models of ZFC, like everything else, live inside V, but they are special because they look a lot like V itself. This is because it turns out that virtually all mathematical proofs of the existence of some object X in V can be mimicked by a proof from the ZFC axioms, thereby proving that any model of ZFC must contain an object that is at least highly analogous to X. It turns out that this "analogue" of X is often *equal* to X, especially when M is a standard transitive model. For example, it turns out that every standard transitive model M of ZFC contains all the von Neumann ordinals as well as $\aleph_0$.

However, the analogue of a mathematical object X is not *always* equal to X. A crucial counterexample is the powerset of $\aleph_0$, denoted by $2^{\aleph_0}$. Naïvely, one might suppose that the powerset axiom of ZFC guarantees that $2^{\aleph_0}$ must be a member of any standard transitive model M. But let us look more closely at the precise statement of the powerset axiom. Given that $\aleph_0$ is in M, the powerset axiom guarantees the existence of y in M with the following property: For every z in M, $z \in y$ if and only if every w in M satisfying $w \in z$ also satisfies $w \in \aleph_0$. Now, does it follow that y is precisely the set of all subsets of $\aleph_0$?

No. First of all, it is not even immediately clear that z is a subset of $\aleph_0$; the axiom does not require that *every* w satisfying $w \in z$ also satisfies $w \in \aleph_0$; it requires only that *every w in M* satisfying $w \in z$ satisfies $w \in \aleph_0$. However, under our assumption that M is *transitive,* every $w \in z$ is in fact in M, so indeed z is a subset of $\aleph_0$.

More importantly, though, y does not contain *every* subset of $\aleph_0$; it contains *only those subsets of* $\aleph_0$ *that are in* M. So if, for example, M happens to be *countable* (i.e., M contains only countably many elements), then y will be countable, and so a fortiori y cannot be equal to $2^{\aleph_0}$, since $2^{\aleph_0}$ is uncountable. The set y, which we might call the *powerset of* $\aleph_0$ *in* M*,* is not the same as the "real" powerset of $\aleph_0$, a.k.a. $2^{\aleph_0}$; many subsets of $\aleph_0$ are "missing" from y.

This is a subtle and important point, so let us explore it further. We may ask, is it really possible for a standard transitive model of ZFC to be countable? Can we not mimic (in ZFC) Cantor's famous proof that $2^{\aleph_0}$ is uncountable to show that M must contain an uncountable set, and then conclude by transitivity that M itself must be uncountable?

The answer is no. Cantor's theorem states that there is no bijection between $\aleph_0$ and $2^{\aleph_0}$. If we carefully mimic Cantor's proof with a proof from the ZFC axioms, then we find that Cantor's theorem tells us that there is indeed a set y in M that plays the role of the powerset of $\aleph_0$ in M, and that there is *no bijection in* M between $\aleph_0$ and y. However, this fact does not mean that there is *no bijection at all* between $\aleph_0$ and y. There might be a bijection *in* V between them; we know only that such a bijection *cannot be a member of* M; it is "missing" from M. So Cantor's theorem does not exclude the possibility that y, as well as M, is countable, even though y is necessarily "uncountable in M."[6] It turns out that something stronger can be said: the so-called *Löwenheim-Skolem theorem* says that if there are any models of ZFC at all, then in fact there exist countable models.

More generally, one says that a concept in V is *absolute* if it coincides with its counterpart in M. For example, "the empty set," "is a member of," "is a subset of,"

[6] This curious state of affairs often goes by the name of *Skolem's paradox.*

"is a bijection," and "$\aleph_0$" all turn out to be absolute for standard transitive models. On the other hand, "is the powerset of" and "uncountable" are *not* absolute. For a concept that is not absolute, we must distinguish carefully between the concept "in the real world" (i.e., in V) and the concept in M.

A careful study of ZFC necessarily requires keeping track of exactly which concepts are absolute and which are not. However, since the majority of basic concepts are absolute, except for those associated with taking powersets and cardinalities, in this paper we will adopt the approach of mentioning non-absoluteness only when it is especially relevant.

5. How one might try to build a model satisfying ¬CH

The somewhat counterintuitive fact that ZFC has countable models with many missing subsets provides a hint as to how one might go about constructing a model for ZFC that satisfies ¬CH. Start with a countable standard transitive model M. The elementary theory of cardinal numbers tells us that there is always a smallest cardinal number after any given cardinal number, so let $\aleph_1, \aleph_2, \ldots$ denote the next largest cardinals after $\aleph_0$. As usual we can mimic the proofs of these facts about cardinal numbers with formal proofs from the axioms of ZFC, to conclude that there is a set in M that plays the role of $\aleph_2$ in M. We denote this set by $\aleph_2^M$. Let us now construct a function F from the Cartesian product $\aleph_2^M \times \aleph_0$ into the set $2 = \{0, 1\}$. We may interpret F as a sequence of functions from $\aleph_0$ into 2. Because M is countable and transitive, so is $\aleph_2^M$; thus we can easily arrange for these functions to be pairwise distinct. Now, if F is already in M, then M satisfies ¬CH! The reason is that functions from $\aleph_0$ into 2 can be identified with subsets of $\aleph_0$, and F therefore shows us that the powerset of $\aleph_0$ in M must be at least $\aleph_2$ in M. Done!

But what if F is missing from M? A natural idea is to add F to M to obtain a larger model of ZFC, that we might call $M[F]$.[7] The hope would be that F can be added in a way that does not "disturb" the structure of M too much, so that the argument in the previous paragraph can be carried over into $M[F]$, which would therefore satisfy ¬CH.

Miraculously, this seemingly naïve idea actually works! There are, of course, numerous technical obstacles to be surmounted, but the basic plan as outlined above is on the right track. For those who like to think algebraically, it is quite appealing to learn that forcing is a technique for constructing new models from old ones by adjoining a new element that is missing from the original model. Even without any further details, one can already imagine that the ability to adjoin new elements to an existing model gives us enormous flexibility in our quest to create models with desired properties. And indeed, this is true; it is the reason why forcing is such a powerful idea.

What technical obstacles need to be surmounted? The first thing to note is that one clearly cannot add only the set F to M and expect to obtain a model of ZFC; one must also add, at minimum, every set that is "constructible" from F together with elements of M, just as when we create an extension of an algebraic

[7] Later on we will use the notation $M[U]$ rather than $M[F]$ because it will turn out to be more natural to think of the larger model as being obtained by adjoining another set U that is closely related to F, rather than by adjoining F itself. For our purposes, $M[U]$ and $M[F]$ can just be thought of as two different names for the same object.

object by adjoining x, we must also adjoin everything that is *generated* by x. We will not define "constructible" precisely here, but it is the same concept that Gödel used to prove that CH is consistent with ZFC, and in particular it was already a familiar concept before Cohen came onto the scene.

A more serious obstacle is that it turns out that we cannot, for example, simply take an arbitrary subset a of $\aleph_0$ that is missing from M and adjoin a, along with everything constructible from a together with elements of M, to M; the result will not necessarily be a model of ZFC. A full explanation of this result would take us too far afield—the interested reader should see page 111 of Cohen's book [**6**]—but the rough idea is that we could perversely choose a to be a set that encodes explicit information about the size of M, so that adjoining a would create a kind of self-referential paradox. Cohen goes on to say:

> Thus a must have certain special properties.... Rather than describe a directly, it is better to examine the various properties of a and determine which are desirable and which are not. The chief point is that we do not wish a to contain "special" information about M, which can only be seen from the outside.... The a which we construct will be referred to as a "generic" set relative to M. The idea is that all the properties of a must be "forced" to hold merely on the basis that a behaves like a "generic" set in M. This concept of deciding when a statement about a is "forced" to hold is the key point of the construction.

Cohen then proceeds to explain the forcing concept, but at this point we will diverge from Cohen's account and pursue instead the concept of a *Boolean-valued model* of ZFC. This approach was developed by Scott, Solovay, and Vopěnka starting in 1965, and in my opinion is the most intuitive way to proceed at this juncture. We will return to Cohen's approach later.

6. Boolean-valued models

To recap, we have reached the point where we see that if we want to construct a model of $\neg$CH, it would be nice to have a method of starting with an arbitrary standard transitive model M of ZFC, and building a new structure by adjoining some subsets that are missing from M. We explain next how this can be done, but instead of giving the construction right away, we will work our way up to it gradually.

6.1. Motivational discussion. Inspired by Cohen's suggestion, we begin by considering *all possible statements* that might be true of our new structure, and then deciding which ones we want to hold and which ones we do not want to hold.

To make the concept of "all possible statements" precise, we must introduce the concept of a *formal language.* Let $\mathfrak{S}$ denote the set of all sentences in the *first-order language of set theory,* i.e., all sentences built out of "atomic" statements such as $x = y$ and xRy (where x and y are *constant symbols* that each represent some fixed element of the domain) using the Boolean connectives OR, AND, and NOT and the quantifiers $\exists$ and $\forall$. The axioms of ZFC can all be expressed in this formal language, as can any theorems (or non-theorems, for that matter) of ZFC. For example, "if A then B" (written $A \to B$) can be expressed as (NOT A) OR B, "A iff B" (written $A \leftrightarrow B$) can be expressed as $(A \to B)$ AND $(B \to A)$, "x is a subset of y" (written $x \subseteq y$) can be expressed as $\forall z\,((zRx) \to (zRy))$, and the powerset axiom can be expressed as

$$\forall x\,\exists y\,\forall z\,((z \subseteq x) \leftrightarrow (zRy)).$$

An important observation is that when choosing which sentences in $\mathfrak{S}$ we want to hold in our new structure, we are subject to certain constraints. For example, if the sentences ϕ and ψ hold, then the sentence ϕ AND ψ must also hold. A natural way to track these constraints is by means of a *Boolean algebra.* The most familiar example of a Boolean algebra is the family 2^S of all subsets of a given set S, partially ordered by inclusion. More generally, a Boolean algebra is any partially ordered set with a minimum element $\mathbf{0}$ and a maximum element $\mathbf{1}$, in which any two elements x and y have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$ (in the example of 2^S, $\vee$ is set union and $\wedge$ is set intersection), where $\vee$ and $\wedge$ distribute over each other (i.e., $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$) and every element x has a *complement,* i.e., an element x^* such that $x \vee x^* = \mathbf{1}$ and $x \wedge x^* = \mathbf{0}$.

There is a natural correspondence between the concepts $\mathbf{0}$, $\mathbf{1}$, $\vee$, $\wedge$, and $*$ in a Boolean algebra and the concepts of falsehood, truth, OR, AND, and NOT in logic. This observation suggests that if we know that we want certain statements of $\mathfrak{S}$ to hold in our new structure but are unsure of others, then we can try to record our state of partial knowledge by picking a suitable Boolean algebra $\mathbb{B}$, and mapping every sentence $\phi \in \mathfrak{S}$ to some element of $\mathbb{B}$ that we denote by $[\![\phi]\!]^{\mathbb{B}}$. If ϕ is "definitely true" then we set $[\![\phi]\!]^{\mathbb{B}} = \mathbf{1}$ and if ϕ is "definitely false" then we set $[\![\phi]\!]^{\mathbb{B}} = \mathbf{0}$; otherwise, $[\![\phi]\!]^{\mathbb{B}}$ takes on some intermediate value between $\mathbf{0}$ and $\mathbf{1}$. In a sense, we are developing a kind of "multi-valued logic" or "fuzzy logic"[8] in which some statements are neither true nor false but lie somewhere in between.

It is clear that the mapping $\phi \mapsto [\![\phi]\!]^{\mathbb{B}}$ should satisfy the conditions

$$[\![\phi \text{ OR } \psi]\!]^{\mathbb{B}} = [\![\phi]\!]^{\mathbb{B}} \vee [\![\psi]\!]^{\mathbb{B}} \tag{6.1}$$

$$[\![\phi \text{ AND } \psi]\!]^{\mathbb{B}} = [\![\phi]\!]^{\mathbb{B}} \wedge [\![\psi]\!]^{\mathbb{B}} \tag{6.2}$$

$$[\![\text{NOT}\,\phi]\!]^{\mathbb{B}} = ([\![\phi]\!]^{\mathbb{B}})^* \tag{6.3}$$

What about atomic expressions such as $[\![x = y]\!]^{\mathbb{B}}$ and $[\![xRy]\!]^{\mathbb{B}}$? Again, if we definitely want certain equalities or membership statements to hold but want to postpone judgment on others, then we are led to the idea of tracking these statements using a structure consisting of "fuzzy sets." To make this precise, let us first observe that an ordinary set may be identified with a function whose range is the trivial Boolean algebra with just two elements $\mathbf{0}$ and $\mathbf{1}$, and that sends the members of the set to $\mathbf{1}$ and the non-members to $\mathbf{0}$. Generalizing, if $\mathbb{B}$ is an arbitrary Boolean algebra, then a "fuzzy set" should take a set of "potential members," which should themselves be fuzzy sets, and assign each potential member y a value in $\mathbb{B}$ corresponding to the "degree" to which y is a member of x. More precisely, we define a a $\mathbb{B}$-*valued set* to be a function from a set of $\mathbb{B}$-valued sets to $\mathbb{B}$. (Defining $\mathbb{B}$-valued sets in terms of $\mathbb{B}$-valued sets might appear circular, but the solution is to note that the empty set is a $\mathbb{B}$-valued set; we can then build up other $\mathbb{B}$-valued sets inductively.)

[8]We use scare quotes as these terms, and the term "fuzzy set" that we use later, have meanings in the literature that are rather different from the ideas that we are trying to convey here.

6.2. Construction of $M^{\mathbb{B}}$. We are now in a position to describe more precisely our plan for constructing a new model of ZFC from a given model M. We pick a suitable Boolean algebra $\mathbb{B}$, and we let $M^{\mathbb{B}}$ be the set of all $\mathbb{B}$-valued sets in M. The set $\mathfrak{S}$ should have one constant symbol for each element of $M^{\mathbb{B}}$. We define a a map $\phi \mapsto [\![\phi]\!]^{\mathbb{B}}$ from $\mathfrak{S}$ to $\mathbb{B}$, which should obey equations such as (6.1)–(6.3) and should send the axioms of ZFC to $\mathbf{1}$. The structure $M^{\mathbb{B}}$ will be a so-called *Boolean-valued model* of ZFC; it will not actually be a model of ZFC, because it will consist of "fuzzy sets" and not sets, and if you pick an arbitrary $\phi \in \mathfrak{S}$ and ask whether it holds in $M^{\mathbb{B}}$, then the answer will often be neither "yes" nor "no" but some element of $\mathbb{B}$ (whereas if N is an actual model of ZFC then either N satisfies ϕ or it doesn't). On the other hand, $M^{\mathbb{B}}$ will satisfy ZFC, in the sense that $[\![\phi]\!]^{\mathbb{B}} = \mathbf{1}$ for every ϕ in ZFC. To turn $M^{\mathbb{B}}$ into an actual model of ZFC with desired properties, we will take a suitable quotient of $M^{\mathbb{B}}$ that eliminates the fuzziness.

We have already started to describe $M^{\mathbb{B}}$ and the map $[\![\cdot]\!]^{\mathbb{B}}$, but we are not done. For example, we need to deal with expressions involving the quantifiers $\exists$ and $\forall$. These may not appear to have a direct counterpart in the formalism of Boolean algebras, but notice that another way to say that there exists x with a certain property is to say that either a has the property or b has the property or c has the property or..., where we enumerate all the entities in the universe one by one. This observation leads us to the definition

$$[\![\exists x\, \phi(x)]\!]^{\mathbb{B}} = \bigvee_{a \in M^{\mathbb{B}}} [\![\phi(a)]\!]^{\mathbb{B}} \tag{6.4}$$

Now there is a potential problem with (6.4): In an arbitrary Boolean algebra, an *infinite* subset of elements may not have a least upper bound, so the right-hand side of (6.4) may not be defined. We solve this problem by fiat: First we define a *complete Boolean algebra* to be a Boolean algebra in which arbitrary subsets of elements have a least upper bound and a greatest lower bound. We then require that $\mathbb{B}$ be a complete Boolean algebra; then (6.4) makes perfect sense, as does the equation

$$[\![\forall x\, \phi(x)]\!]^{\mathbb{B}} = \bigwedge_{a \in M^{\mathbb{B}}} [\![\phi(a)]\!]^{\mathbb{B}} \tag{6.5}$$

Equations (6.4) and (6.5) take care of $\exists$ and $\forall$, but we have still not defined $[\![xRy]\!]^{\mathbb{B}}$ or $[\![x = y]\!]^{\mathbb{B}}$, or ensured that $M^{\mathbb{B}}$ satisfies ZFC. The definitions of $[\![x = y]\!]^{\mathbb{B}}$ and $[\![xRy]\!]^{\mathbb{B}}$ are surprisingly delicate; there are many plausible attempts that fail for subtle reasons. The impatient reader can safely skim the details in the next paragraph and just accept the final equations (6.6)–(6.7).

We follow the treatment on pages 22–23 in Bell's book [**3**], which motivates the definitions of $[\![x = y]\!]^{\mathbb{B}}$ and $[\![xRy]\!]^{\mathbb{B}}$ by listing several equations that one would like to hold and inferring what the definitions "must" be. First, we want the axiom of extensionality to hold in $M^{\mathbb{B}}$; this suggests the equation

$$[\![x = y]\!]^{\mathbb{B}} = [\![(\forall w\,(wRx \to wRy)) \text{ AND } (\forall w\,(wRy \to wRx))]\!]^{\mathbb{B}}.$$

Another plausible equation is

$$[\![xRy]\!]^{\mathbb{B}} = [\![\exists w\,((wRy) \text{ AND } (w = x))]\!]^{\mathbb{B}}.$$

It is also plausible that the expression $[\![\exists w\,((wRy) \text{ AND } \phi(w))]\!]^{\mathbb{B}}$ should depend only on the values of $[\![\phi(w)]\!]^{\mathbb{B}}$ for those w that are actually in the domain of y

(recall that y, being a $\mathbb{B}$-valued set, is a function from a set of $\mathbb{B}$-valued sets to $\mathbb{B}$, and thus has a domain $\operatorname{dom}(y)$). Also, the value of $[\![wRy]\!]^{\mathbb{B}}$ should be closely related to the value of $y(w)$. We are thus led to the equations

$$[\![\exists w\,(wRy \text{ AND } \phi(w))]\!]^{\mathbb{B}} = \bigvee_{w\in\operatorname{dom}(y)} \big(y(w) \wedge [\![\phi(w)]\!]^{\mathbb{B}}\big)$$

$$[\![\forall w\,(wRy \to \phi(w))]\!]^{\mathbb{B}} = \bigwedge_{w\in\operatorname{dom}(y)} \big(y(w) \Rightarrow [\![\phi(w)]\!]^{\mathbb{B}}\big)$$

where $x \Rightarrow y$ is another way of writing $x^* \vee y$. All these equations drive us to the definitions

$$[\![xRy]\!]^{\mathbb{B}} = \bigvee_{w\in\operatorname{dom}(y)} \big(y(w) \wedge [\![x = w]\!]^{\mathbb{B}}\big) \tag{6.6}$$

$$[\![x = y]\!]^{\mathbb{B}} = \bigwedge_{w\in\operatorname{dom}(x)} \big(x(w) \Rightarrow [\![wRy]\!]^{\mathbb{B}}\big) \wedge \bigwedge_{w\in\operatorname{dom}(y)} \big(y(w) \Rightarrow [\![wRx]\!]^{\mathbb{B}}\big) \tag{6.7}$$

The definitions (6.6) and (6.7) again appear circular, because they define $[\![x = y]\!]^{\mathbb{B}}$ and $[\![xRy]\!]^{\mathbb{B}}$ in terms of each other, but again (6.6) and (6.7) should be read as a joint inductive definition.

One final remark is needed regarding the definition of $M^{\mathbb{B}}$. So far we have not imposed any constraints on $\mathbb{B}$ other than that it be a complete Boolean algebra. But without some such constraints, there is no guarantee that $M^{\mathbb{B}}$ will satisfy ZFC. For example, let us see what happens with the powerset axiom. Given x in $M^{\mathbb{B}}$, it is natural to construct the powerset y of x in $M^{\mathbb{B}}$ by letting

$$\operatorname{dom}(y) = \mathbb{B}^{\operatorname{dom}(x)},$$

i.e., the "potential members" of y should be precisely the maps from $\operatorname{dom}(x)$ to $\mathbb{B}$. Moreover, for each $w \in \operatorname{dom}(y)$, the value of $y(w)$ should be $[\![w \subseteq x]\!]^{\mathbb{B}}$. The catch is that if $\mathbb{B}$ is not in M, then maps from $\operatorname{dom}(x)$ to $\mathbb{B}$ may not be $\mathbb{B}$-valued sets in M. The simplest way out of this difficulty is to require that $\mathbb{B}$ be in M, and we shall indeed require this.[9] Once we impose this condition, we can weaken the requirement that $\mathbb{B}$ be a complete Boolean algebra to the requirement that $\mathbb{B}$ be a complete Boolean algebra *in M,* meaning that infinite least upper bounds and greatest lower bounds *over subsets of B that are in M* are guaranteed to exist, but not necessarily in general. ("Complete," being related to taking powersets, is not absolute.) Examination of the definitions of $M^{\mathbb{B}}$ and $[\![\cdot]\!]^{\mathbb{B}}$ reveals that $\mathbb{B}$ only needs to be a complete Boolean algebra in M, and it turns out that this increased flexibility in the choice of $\mathbb{B}$ is very important.

We are now done with the definition of the Boolean-valued model $M^{\mathbb{B}}$. To summarize, we pick a Boolean algebra $\mathbb{B}$ in M that is complete in M, let $M^{\mathbb{B}}$ be the set of all $\mathbb{B}$-valued sets in M, and define $[\![\cdot]\!]^{\mathbb{B}}$ using equations (6.1)–(6.7).

At this point, one needs to perform a long verification that $M^{\mathbb{B}}$ satisfies ZFC, and that the rules of logical inference behave as expected in $M^{\mathbb{B}}$ (so that, for example, if $[\![\phi]\!]^{\mathbb{B}} = \mathbf{1}$ and ψ is a logical consequence of ϕ then $[\![\psi]\!]^{\mathbb{B}} = \mathbf{1}$). We omit these details because they are covered well in Bell's book [**3**]. Usually, as in the case

[9] While choosing $\mathbb{B}$ to be in M suffices to make everything work, it is not strictly necessary. *Class forcing* involves certain carefully constructed Boolean algebras $\mathbb{B}$ that are not in M. However, this is an advanced topic that is not needed for proving the independence of CH.

of the powerset axiom above, it is not too hard to guess how to construct the object whose existence is asserted by the ZFC axiom, using the fact that M satisfies ZFC, although in some cases, completing the argument in detail can be tricky.

6.3. Modding out by an ultrafilter. As we stated above, the way to convert our Boolean-valued model $M^{\mathbb{B}}$ to an actual model of ZFC is to take a suitable quotient. That is, we need to pick out precisely the statements that are true in our new model. To do this, we choose a subset U of $\mathbb{B}$ that contains $[\![\phi]\!]^{\mathbb{B}}$ for every statement ϕ that holds in the new model of ZFC. The set U, being a "truth definition" for our new model, has to have certain properties; for example, since for every ϕ, either ϕ or NOT ϕ must hold in the new model, it follows that for all x in $\mathbb{B}$, U must contain either x or x^*. Similarly, thinking of membership in U as representing "truth," we see that U should have the following properties:

(1) $1 \in U$;
(2) $0 \notin U$;
(3) if $x \in U$ and $y \in U$ then $x \wedge y \in U$;
(4) if $x \in U$ and $x \leq y$ (i.e., $x \wedge y = x$) then $y \in U$;
(5) for all x in $\mathbb{B}$, either $x \in U$ or $x^* \in U$.

A subset U of a Boolean algebra having the above properties is called an *ultrafilter.*

Given any ultrafilter U in $\mathbb{B}$ (U does not have to be in M), we define the quotient $M^{\mathbb{B}}/U$ as follows. The elements of $M^{\mathbb{B}}/U$ are equivalence classes of elements of $M^{\mathbb{B}}$ under the equivalence relation

$$x \sim_U y \quad \text{iff} \quad [\![x = y]\!]^{\mathbb{B}} \in U.$$

If we write x^U for the equivalence class of x, then the binary relation of $M^{\mathbb{B}}/U$—which we shall denote by the symbol $\in_U$—is defined by

$$x^U \in_U y^U \quad \text{iff} \quad [\![xRy]\!]^{\mathbb{B}} \in U.$$

It is now fairly straightforward to verify that $M^{\mathbb{B}}/U$ is a model of ZFC; the hard work has already been done in verifying that $M^{\mathbb{B}}$ satisfies ZFC.

7. Generic ultrafilters and the conclusion of the proof sketch

At this point we have a powerful theorem in hand. We can take any model M, any complete Boolean algebra $\mathbb{B}$ in M, and any ultrafilter U of M, and form a new model $M^{\mathbb{B}}/U$ of ZFC. We can now experiment with various choices of M, $\mathbb{B}$, and U to construct all kinds of models of ZFC with various properties.

So let us revisit our plan (in Section 5) of starting with a standard transitive model and inserting some missing subsets to obtain a larger standard transitive model. If we try to use our newly constructed machinery to carry out this plan, then we soon find that $M^{\mathbb{B}}/U$ need not, in general, be (isomorphic to) a standard transitive model of ZFC, even if M is. Some extra conditions need to be imposed.

Cohen's insight—perhaps his most important and ingenious one—is that in many cases, including the case of CH, the right thing to do is to require that U be *generic.* The term "generic" can be defined more generally in the context of an arbitrary partially ordered set P. First define a subset D of P to be *dense* if for all p in P, there exists q in D such that $q \leq p$. Then a subset of P is *generic* if it intersects every dense subset. In our current setting, the partially ordered set P is $\mathbb{B}\backslash\{\mathbf{0}\}$, and the crucial condition on U is that it be *M-generic* (or *generic over M*), meaning that U intersects every dense subset $D \subseteq (\mathbb{B}\backslash\{\mathbf{0}\})$ that is a member of M.

If U is M-generic, then $M^{\mathbb{B}}/U$ has many nice properties; it is (isomorphic to) a standard transitive model of ZFC, and equally importantly, it contains U. In fact, if U is M-generic, then $M^{\mathbb{B}}/U$ is the smallest standard transitive model of ZFC that contains both M and U. For this reason, when U is M-generic, one typically writes $M[U]$ instead of $M^{\mathbb{B}}/U$. We have realized the dream of adjoining a new subset of M to obtain a larger model (remember that U is a subset of $\mathbb{B}$ and we have required $\mathbb{B}$ to be in M).

It is, of course, not clear that M-generic ultrafilters exist in general. However, if M is countable, then it turns out to be easy to prove the existence of M-generic ultrafilters; essentially, one just lists the dense sets and hits them one by one. If M is uncountable then the Boolean-valued model machinery still works fine, but M-generic ultrafilters may not exist.[10] Fortunately for us, the idea sketched at the beginning of Section 5 relies on M being countable anyway.

Let us now return to that idea and complete the proof sketch. Start with a countable standard transitive model M of ZFC. If M does not already satisfy $\neg$CH, then let P be the partially ordered set of all *finite partial functions* from $\aleph_2^M \times \aleph_0$ into 2, partially ordered by *reverse* inclusion. (A finite partial function is a finite set of ordered pairs whose first coordinate is in the domain and whose second coordinate is in the range, with the property that no two ordered pairs have the same first element.) There is a standard method, which we shall not go into here, of *completing* an arbitrary partially ordered set to a complete Boolean algebra; we take the completion of P in M to be our Boolean algebra $\mathbb{B}$. Now take an M-generic ultrafilter U, which exists because M is countable. If we blur the distinction between P and its completion $\mathbb{B}$ for a moment, then we claim that $F := \bigcup U$ is a partial function from $\aleph_2^M \times \aleph_0$ to 2. To check this, we just need to check that any two elements x and y of U are consistent with each other where they are both defined, but this is easy: Since U is an ultrafilter, x and y have a common lower bound z in U, and both x and y are consistent with z. Moreover, F is a *total* function; this is because U is generic, and the finite partial functions that are defined at a specified point in the domain form a dense set (we can extend any partial function by defining it at that point if it is not defined already). Also, the sequence of functions from $\aleph_0$ to 2 encoded by F are pairwise distinct; again this is because U is generic, and the condition of being pairwise distinct is a dense condition. The axioms of ZFC ensure that $F \in M[U]$, so $M[U]$ gives us the desired model of $\neg$CH.

There is one important point that we have swept under the rug in the above proof sketch. The set $\aleph_2^M$ is still hanging around in $M[U]$, but it is conceivable that $\aleph_2^M$ may no longer play the role of $\aleph_2$ in $M[U]$; i.e., it may be that $\aleph_2^M \neq \aleph_2^{M[U]}$. Cardinalities are not absolute, and so *cardinal collapse* can occur, i.e., the object that plays the role of a particular cardinal number in M may not play that same role in an extension of M. In fact, cardinal collapse does not occur in this particular case but this fact must be checked.[11] We omit the details, since they are covered thoroughly in textbooks.

[10] This limitation of uncountable models is not a big issue in practice, because typically the Löwenheim-Skolem theorem allows us to replace an uncountable model with a countable surrogate.

[11] The fact that cardinals do not collapse here can be traced to the fact that the Boolean algebra in question satisfies a combinatorial condition called the *countable chain condition* in M.

8. But wait—what about forcing?

The reader may be surprised—justifiably so—that we have come to the end of our proof sketch without ever precisely defining *forcing.* Does "forcing" not have a precise technical meaning?

Indeed, it does. In Cohen's original approach, he asked the following fundamental question. Suppose that we want to adjoin a "generic" set U to M. What properties of the new model will be "forced" to hold if we have only partial information about U, namely we know that some element p of M is in U?

If we are armed with the machinery of Boolean-valued models, then we can answer Cohen's question. Let us informally say that p *forces* ϕ (written $p \Vdash \phi$) if for every M-generic ultrafilter U, ϕ must hold in $M[U]$ whenever $p \in U$. Note that U plays two roles simultaneously; it *is* the generic set that we are adjoining to M, and it also picks out the true statements in $M[U]$. By the definition of an ultrafilter, we see that if $p \le [\![\phi]\!]^{\mathbb{B}}$, then ϕ must be true in $M[U]$ if $p \in U$. Therefore we can give the following formal definition of "$p \Vdash \phi$":

$$p \Vdash \phi \quad \text{iff} \quad p \le [\![\phi]\!]^{\mathbb{B}}. \tag{8.1}$$

The simplicity of equation (8.1) explains why our proof sketch did not need to refer to forcing explicitly. Forcing is actually implicit in the proof, but since $\Vdash$ has such a simple definition in terms of $[\![\cdot]\!]^{\mathbb{B}}$, it is possible in principle to produce a proof of Cohen's result without explicitly using the symbol $\Vdash$ at all, referring only to $[\![\cdot]\!]^{\mathbb{B}}$ and Boolean algebra operations.

Of course, Cohen did not have the machinery of Boolean-valued models available. What he did was to figure out what properties the expression $p \Vdash \phi$ ought to have, given that one is trying to capture the notion of the logical implications of knowing that p is a member of our new "generic" set. For example, one should have $p \Vdash (\phi \text{ AND } \psi)$ iff $p \Vdash \phi$ and $p \Vdash \psi$, by the following reasoning: If we know that membership of p in U forces ϕ to hold and it also forces ψ to hold, then membership of p in U must also force ϕ AND ψ to hold.

By similar but more complicated reasoning, Cohen devised a list of rules analogous to (6.1)–(6.7) that he used to define $p \Vdash \phi$ for any statement ϕ in $\mathfrak{S}$. In this way, he built all the necessary machinery on the basis of the forcing relation, without ever having to introduce Boolean algebras.

Thus there are (at least) two different ways to approach this subject, depending on whether $\Vdash$ or $[\![\cdot]\!]^{\mathbb{B}}$ is taken to be the fundamental concept. For many applications, these two approaches ultimately amount to almost the same thing, since we can use equation (8.1) to pass between them. In this paper I have chosen the approach using Boolean-valued models because I feel that the introduction of "fuzzy sets" and the Boolean algebra $\mathbb{B}$ are relatively easy to motivate. In Cohen's approach one still needs to introduce at some point some "fuzzy sets" (called *names* or *labels*) and a partial order, and these seem (to me at least) to be pulled out of a hat. Also, the definitions (6.1)–(6.7) are somewhat simpler than the corresponding definitions for $\Vdash$.

On the other hand, even when one works with Boolean-valued models, Cohen's intuition about generic sets U, and what is forced to be true if we know that $p \in U$, is often extremely helpful. For example, recall from our proof sketch that the constructed functions from $\aleph_0$ to 2 were pairwise distinct. In geometry, "generically" chosen functions will not be equal; distinctness is a dense condition. Cohen's intuition thus leads us to the (correct) expectation that our generically chosen functions will also be distinct, because no finite $p \in U$ can force two of them to be equal. This kind of reasoning is invaluable in more complicated applications.

9. Final remarks

We should mention that the Boolean-valued-model approach has some disadvantages. For example, set theorists sometimes find the need to work with models of axioms that do not include the powerset axiom, and then the Boolean-valued model approach does not work, because "complete" does not really make sense in such contexts. Also, Cohen's original approach allows one to work directly with an arbitrary partially ordered set P that is not necessarily a Boolean algebra, and a *generic filter* rather than a generic ultrafilter. (A subset F of P is a *filter* if $p \in F$ and $p \leq q$ implies $q \in F$, and every p and q have a common lower bound in F.) In our proof sketch we have already caught a whiff of the fact that in many cases, there is some partially ordered set P lying around that captures the combinatorics of what is really going on, and having to complete P to a Boolean algebra is a technical nuisance; it is much more convenient to work with P directly. If the reader prefers this approach, then Kunen's book [**10**] would be my recommended reference. Note that Kunen helpfully supplements his treatment with an abbreviated discussion of Boolean-valued models and the relationship between the two different approaches.

In my opinion, the weakest part of the exposition in this paper is the treatment of *genericity,* whose definition appears to come out of nowhere. A posteriori one can see that the definition works beautifully, but how would one guess a priori that the geometric concepts of dense sets and generic sets would be so apropos in this context, and come up with the right precise definitions? Perhaps the answer is just that Cohen was a genius, but perhaps there is a better approach yet to be discovered that will make it all clear.

Let us conclude with some suggestions for further reading. Easwaran [**8**] and Wolf [**16**] give very nice overviews of forcing written in the same spirit as the present paper, giving details that are critical for understanding but omitting messy technicalities. Scott's paper [**14**] is a classic exposition written for non-specialists, and Cohen [**7**] gave a lecture late in his life about how he discovered forcing. The reader may also find it helpful to study the connections that forcing has with topology [**5**], topos theory [**12**], modal logic [**15**], arithmetic [**4**], proof theory [**1**] and computational complexity [**9**]. It may be that insights from these differing perspectives can be synthesized to solve the open exposition problem of forcing.

10. Acknowledgments

This paper grew out of an article entitled "Forcing for dummies" that I posted to the USENET newsgroup `sci.math.research` in 2001. However, in that article I did not employ the Boolean-valued model approach, and hence the line of exposition was quite different. Interested readers can easily find the earlier version on the web.

I thank Costas Drossos, Ali Enayat, Martin Goldstern, and Matt Yurkewych for pointing me to references [**14**], [**5**], [**7**], and [**15**] respectively.

I am deeply indebted to Matthew Wiener and especially Andreas Blass for their many prompt and patient answers to my dumb questions about forcing.

Thanks also to Fred Kochman, Miller Maley, Christoph Weiss, Steven Gubkin, James Hirschhorn, and the referee for comments on an earlier draft of this paper that have improved the exposition and fixed some bugs.

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