Baye's Theorem

Assume that you know the probability that a child will be born with blond hair given that both his parents have blond hair. You might also be interested in knowing the probability that a child's parents both have blond hair given that the child has blond hair.

Stated more formally, if you preform and experiment with possible outcomes E and F we are interested in how the conditional probabilities $P(E|F)$ and $P(F|E)$ might relate to each other. This question is answered by *Baye's Theorem*.

Baye's Theorem Let E and F be two possible events of an experiment, then

$$P(F|E) = \frac{P(F) \cdot P(E|F)}{P(F) \cdot P(E|F) + P(F') \cdot P(E|F')}.$$

Question: Where does this formula come from?

Helful Hint: If the form that Baye's theorem is stated above is too hard to remeber. Just remember the following

$$P(F|E) = \frac{P(E \cap F)}{P(E)}.$$

We'll see in a few pages how this will be much more helpfull than memorizing Baye's theorem.

For now, we have to know how to use Baye's theorem. The best way to compute the probabilities that you need for Baye's theorem is to us a probability tree. Let's look at some examples to refresh our memory.

Exercise: Assume that you have an urn with 2 red balls and 4 blue balls. Draw a probability tree detailing the possible combinations of balls that we can get when we pick out two balls.

Exercise (General Probability Tree): Assume that an experiment has outcomes E and F. Draw a probability tree that details all the possibilities if we run the same experiment twice.

Exercise: A study has found that the probability that the first born child of an expecting mother is a boy is .5. Once a mother has a first child, her second child is more likely to be of the same gender as the first. In particular, given that a mother has had a boy as her first child, the probability that her second child will be a boy is 57%. If the mother had a girl as her first child, then the probability that the second child will be a girl is 55%.

(a) Draw a probability tree detailing all the possible combinations the first two children that a mother can have given the information above.

(b) What are the first two children of a family most likely to be.

(c) $P(\text{1st child is G}|\text{2nd child is B})$.

(d) $P(\text{2nd child is G})$

(e) $P(\text{1st child is B}|\text{2nd child is B})$.

(f) $P(\text{1st child is B}|\text{2nd child is G})$.

The probability trees that we have drawn are only good if the outcome of an experiement only has two possibilities. Usually, this will not be the case. For example, in the exercise above we assumed that an expecting mother can only have a boy or a girl, but we may also consider the case that she has twins. We will get to this example later, first lets consider how to generalize what we have just done.

Exercise: Assume that a first experiment has three possible outcomes $\{F_1, F_2, F_3\}$ and a second experiment has four possible outcomes $\{E_1, E_2, E_3, E_4\}$. Draw the probability tree that details all the possible combinations of outcomes from the two experiments.

Exercise: Assume that an urn has 1 red ball and 2 blue balls. When you pick out a ball from the urn you keep that ball and add 2 more red balls and 2 more white balls to the urn. Then you pick out a second ball from the urn.

(a) Draw a probability tree that details all the possible combinations of balls that you may have picked out.

(b) Calculate $P(\text{2n ball R})$.

(c) Calculate $P(\text{1st ball R})$.

(d) Calculate $P(\text{2nd ball W})$.

(e) Calculate $P(\text{1s ball B}|\text{2nd ball W})$.

(f) Calculate $P(\text{1s ball R}|\text{2nd ball R})$.

(g) Calculate $P(\text{1s ball R}|\text{2nd ball B})$.

(h) Calculate $P(\text{1s ball B}|\text{2nd ball B})$.

We can formalize what we have just done with a general version of Baye's theorem.

Baye's Theorem (General Version) Let $\{F_1, F_2, \ldots, F_n\}$ be *mutullay exclusive* events then for any event E

$$P(F_i|E) = \frac{P(E|F_i)}{P(F_1)P(E|F_1) + P(F_2)P(E|F_2) + \cdots + P(F_n)P(E|F_n)}.$$

Exercise: A new (and improved) study refutes the findings of the study mentioned earlier. It has found that a mother who is expecting for the first time has a 45% chance of having a boy, 45% change of having a girl and a 10% chance of having a pair of twins. Given that the woman had a boy, she has a 49% chanve of having a second boy, a 43% chance of having a girl and a 8% chance of having a pair of twins. If the woman has a girl as her first child, then the probability that her second child will be a boy is .42, the probability that the second child will be a girl is .48 and the probabilitity that she will have a pair of twins is .1. Finally, if she has a pair of twins as her first offspring, then the second offspring can be a girl, a boy or another pair of twins all with equal chance.

(a) Draw a probability tree describing all the possible combinations of the first two offspring that a mother can have.

(b) Calculate P(2nd offspring is G).

(c) Calculate P(1st offspring is B | 2nd offspring is T).

(c) Calculate P(1st offspring is G | 2nd offspring is B).

(c) Calculate P(1st offspring is B | 2nd offspring is G).

(c) Calculate P(1st offspring is T | 2nd offspring is T).