Solutions of Linear Systems by the Gauss-Jordan Method

The Gauss Jordan method allows us to isolate the coefficients of a system of linear equations making it simpler to solve for.

Creating the Augmented Matrix
To isolate the coefficients of a system of linear equations we create an augmented matrix as follows:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_2y + c_3z = d_3 \end{cases} \quad \text{becomes} \quad \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right].$$

Exercise Represent the following systems of linear equations by an augmented matrix:

$$(1) \begin{cases} x + z = 6 \\ x + 2z + 6y = 9 \\ x - 4y - z = 5 \end{cases}$$

$$(2) \begin{cases} 2x + 3z = y \\ 5x + z = 3 + 2y \\ x + 6y + z = 9 \end{cases}$$

Exercise Construct the system of linear equations from the augmented matricies

$$(1) \left[\begin{array}{cc|c} 2 & 0 & -1 \\ -3 & 9 & 2 \\ 2 & 4 & 0 \end{array}\right].$$

$$(2) \left[\begin{array}{ccc|c} 1 & -1 & 0 & 7 \\ 4 & -3 & 2 & 1 \end{array}\right].$$

Row Operations
We can manipulate an augmented matrix using the following rules:

(1) Switch any two rows.

(2) Multiply a row by a nonzero real number.

(3) Adding a nonzero multiple of one row to any other row.

Exercise Rewrite the given augmented matrix according to the row operations specified.

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ -3 & 2 & 1 & 3 \\ 1 & 0 & -1 & 2 \end{array}\right].$$

(1) $R_1 + 2R_2 \rightarrow R_1$

(2) Switch first and second row, and $R_1 - 2R_3 \rightarrow R_3$.

Gauss Jordan Method

(1) Write system of equations so that variables are on the right side of the equals sign.

(2) Write the augmented matrix for the system of equations

(3) Use the row operations to rewrite the augmented matrix so that the first row looks like:

$$[1\ \ 0\ \ 0\ \ \cdots\ \ 0\ \mid\ a_1]$$

(3) Use the row operations to rewrite the augmented matrix so that the second row looks like:

$$[0\ \ 1\ \ 0\cdots\ \ 0\ \mid\ a_1]$$

(4) Continue this process for as long as you can.

Example Solve the following system of linear equations using the Gauss Jordan method.

$$\begin{cases} 3x + 5y = z \\ 4x - z = 1 - 2y \\ 7x + 4y + z = 1 \end{cases}$$

Check Your Answers !!!
Insert the values that you calculated for x, y and z to check that the system of linear euqations hold.

Solving a system of equations with an infinite number of solutions

Example Solve the following system of linear equations using the Gauss Jordan method.

$$\begin{cases} 4x - 3y + z = 21 - w \\ -2x - y + 2z + 7w = 2 \\ 10x = 15 + 5z + 20w \end{cases}$$

Solution:

What does this mean?

Addition and Subtraction of Matricies

Definition We say that the following matrix is an $n \times m$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

Describe the following matrices.

(1) $\begin{bmatrix} 2 & -5 & 6 \\ 0 & 4 & 0 \\ 9 & 1 & -4 \end{bmatrix}$ (2) $\begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

(3) $\begin{bmatrix} 5 & -4 \\ 0 & 0 \\ 4 & -1 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -1 & 0 & 8 & 5 \end{bmatrix}$

Definiton Two matricies are equal if they are of the same size and have similar coresponding elements.

Example Do there exist values that make the following matricies equal? If so, what are the values? If not, why not?

(1) $\begin{bmatrix} 3 & 7 & 0 & 1 \\ 4 & 3 & -1 & 0 \\ 5 & 6 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} x & 7 & 0 & 1 \\ 4 & y & -1 & z \\ 5 & 6 & 1 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & x \\ 3 & y & 1 \\ 3 & -1 & 1 \end{bmatrix}$

Adding Matrices
To add two matricies which have the *same size* you do the following operation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ . & & & . \\ . & & & . \\ . & & & . \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{bmatrix}.$$

YOU CAN ONLY ADD MATRICES OF THE SAME SIZE!

Notation:
If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ . & & & . \\ . & & & . \\ . & & & . \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{bmatrix}.$$

Multiplying a matrix by a constant: If A is the matrix above and k is a constant then

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1m} \\ ka_{21} & ka_{22} & \cdots & ka_{2m} \\ . & & & . \\ . & & & . \\ . & & & . \\ ka_{n1} & ka_{n2} & \cdots & ka_{nm} \end{bmatrix}$$

Subtracting Matrices
Let A and B be the matrices describes above, then

$$A - B = A + (-B).$$

Examples Preform the Following operations if they are possible. If it is not possible, please explain why.

(1) $\begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} =$

(2) $\begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 & 2 & 2 \end{bmatrix} =$

(4) $-2\begin{bmatrix} 3 & 0 & 1 \\ 0 & -4 & 1 \\ 5 & 2 & 1 \end{bmatrix} =$

(3) $\begin{bmatrix} 3 & 0 & 1 \\ 0 & -4 & 1 \\ 5 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 8 & 10 & 1 \end{bmatrix} =$

(4) $\begin{bmatrix} 2 & -1 & 1 \\ 10 & 4 & 1 \\ 8 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 5 & -7 \\ 1 & 12 & 0 \end{bmatrix} =$

Definition The zero matrix is any matrix with all its entries equal to 0.

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Definition The additive inverse of a matrix A, is the matrix B, such that

$$A + B = \mathbf{0}.$$

NOTE: This implies that the additive inverse of A is $-A$.
Exercise Find the additive inverses of the following matricies.

(1) $\begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix}$

(2) $2\begin{bmatrix} 1 & 0 & 3 \\ 3 & 5 & -1 \end{bmatrix}$

(3) $-\begin{bmatrix} 2 & 5 \\ 0 & 3 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}$