Multiplication of Matricies

Let A and B be the following matricies.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

The product AB is calculated as follows

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

Exercies Take the following products if possible.

(1) $\begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 3 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 5 & -2 \\ 0 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$

(4) $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 4 & 1 \end{bmatrix}$

Question Write down the sizes of the matricies above and the size of thier product. What do you observe?

(1)

(2)

(3)

General Statement If I multiply a $n \times m$ matrix by a $m \times t$ matrix. What is the size of the product?

Exercise Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Take the following products

(1) AB

(2) BA

Question Do we get the same result?

In general, if A and B are matricies, it is NOT ALWAYS TRUE that $AB = BA$!!!

Matrix Inverses

Definition The *identity matrix* is any $n \times n$ matrix with 1's along the diagonal and 0 everywhere else. It is usually denoted by I. For example, the 3×3 identity matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For any matrix A we have that

$$AI = A \quad \text{and} \quad IA = A$$

Check this statement by computing the following products

(1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Definition Let A be a matrix. If there exists a matrix B such that

$$AB = I \quad \text{and} \quad BA = I$$

then B is called the *multiplicative inverse* of A and is denote by A^{-1}.

How to find the multiplicative inverse matrix
Let A be a matrix, in order to find A^{-1} *(if it exists)* we do the following:

(1) Create the augmented matrix $[A|I]$

(2) Use row operations to rewrite your augmented matrix as $[I|B]$

(3) B is the inverse matrix of A.

Exercise Find the inverse matricies of the following matricies if they exist.

(1) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Consider the following system of linear equations.

$$\begin{cases} 2x + 1y &= 3 \\ -1x + 5y + 3z &= 4 \\ 3z + 7y + 9z &= 0 \end{cases}$$

Check that the product of the the following matrices represents the system of linear equations given above.

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 5 & 3 \\ 3 & 7 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

Solving a system of linear equations using matrix multiplication
When we solve for a system of linear equations

$$\begin{cases} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{cases}$$

we want to figure out what x and y equal. We can do this with matricies as follows. Let A, X and B be the following matricies

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

In order to know what x and y are we want to solve for the matrix X. In other word, we want to preform the following operation

$$X = A^{-1}B \tag{1}$$

Question Is it OK if I solve for X by $X = BA^{-1}$?

Why?

Exercise Solve the following system of linear equations using matrix multiplication.

(1) $\begin{cases} x + 3y &= -14 \\ 2x - y &= 7 \end{cases}$

(2) $\begin{cases} 2x + 4y + 6z &= 4 \\ -x - 4y - 3z &= 8 \\ y - z &= -4 \end{cases}$

Question What happens if we cannot find an inverse matrix when using matrix multiplication to solve for a system of linear equations?