Counting Principles

Assume that I have eleven *distinguishable* beads, in the sense that no two beads are the same. I want to count the total number of different necklaces that I can make with these eleven beads. When I begin to make the necklace I notice that there are

11 ways to choose the 1^{st} bead.
10 ways to choose the 2^{nd} bead.
9 ways to choose the 3^{rd} bead.
.
.
.
2 ways to choose the 10^{th} bead.
1 way to choose the 11^{th} bead.

In particular, there are $11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 39916800$ ways to make a necklace with the 11 distinguishable beads.

In general, if I have to make k choices and there are

m_1 ways to make the 1^{st} choice.
m_2 ways to choose the 2^{nd} choice.
m_3 ways to choose the 3^{rd} choice.
.
.
.
m_k ways to choose the 10^{th} choice.

Then the total number of ways that I can pick k things is $m_1 \cdot m_2 \cdots m_{k-1} \cdot m_k$.

Example Suppose that I have two urns. The first urn contains balls numbered 1 through 4 and the second urn contains balls numbered 5 through 7. Say I must pick a ball from the first urn and then a ball from the second urn. How many different ways can I make these picks.

(a) Draw a tree that shows all the possible picks that I can make.

(b) Check that the formula above gives the same number of items that you constructed part (a).

Exercise: A lottery ticket has a 9 digit number on it where each digit can range from 0 to 9. How many possible *distinct* lottery tickets are there?

Exercise: A class of 10 people must a elect a committee consisting of a president, vice president and treasurer. Assuming all the members of the committee must be from the class and that no person may hold more than one position, how many different committees can there be?

Exercise Nicole has a closet with 5 jackets, 10 shirts, 3 pairs of pants and 10 pairs of shoes. An outfit consists of a jacket, shirt, pants and shoes. How many possible outfits can Nicole make from her closet.

Exercise How many ways can we arrange the letters in the alphabet?

Definition: For any whole number n

$$n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)$$

with the convention that $0! = 1$.

Exercise: Make the following computations.

(a) $4!$

(b) $\frac{10!}{(5!)(3!)}$

(c) $\frac{n!}{k!}$ if $k \leq n$.

Exercise Use factorial notation for the following questions.

(a) How many ways can I pick three different numbers from the set $\{2, 5, 6, 3, 8, 7, 1\}$?

(b) How many ways can we arrange the letters in the alphabet?

(c) How many ways can I pick k balls from an urn with n distinguishable balls?

(d) How many ways can I drop 3 indistinguishable balls into 10 distinguishable boxes?

Definition: Given a set with n elements, a *permutation of k elements* are a set of k elements taken from the original n elements with a specified ordering.

Examples:

(1) $(5, 9, 1)$ is a permutation of 3 elements from the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(2) $(\spadesuit, \circledast, \clubsuit, \bigstar)$ is a permutation of 4 element from the set $B = \{\boxplus, \circledast, \circledcirc, \bigstar, \clubsuit, \spadesuit\}$

(3) $(\boxplus, \circledast, \spadesuit)$ and $(\spadesuit, \boxplus, \circledast)$ are *not* the same permutation of 3 elements from the set B.

(4) $(2, 4, 9, 6)$ and $(4, 6, 9, 2)$ are *not* the same permutation of 4 elements from set A.

Questions Answer the following questions using factorial notation.

(a) How many permutations of 3 digits are there from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

(b) How many permutations of 5 elements are there from a set of 11 elements?

(c) How many permutations of k elements are there from a set of n elements?

Definition: We denote by $P(n,k)$ the total number of permutations of k elements from a set of n elements. Furthermore

$$P(n,k) = \frac{n!}{(n-k)!}.$$

Question: How many different bracelets of 5 beads can I make from a total of 11 beads?

Question:

(a) How many distinct 5 digit lottery ticket numbers are there if each digit is from 0 to 9?

(b) How many distinct 5 digit lottery ticket numbers are there if each digit is from 0 to 9 *and* no digit is allowed to repeat?

Question: Assume that an urn has seven balls numbered 1 through 4.

(a) How many ways can I choose two balls if the order in which I choose them matters?

(b) Use a tree to draw all the possible choices choices of two balls that I can make if order matters.

(c) How many ways can I choose two balls if order does not matter?

NOTE: A permutation takes into account the order that the elements are arranged in!

Sometimes sets have elements that are not all distinguishable but not all the same. For example, the set

$$S = \{\boxplus, \bigstar, \circledast, \bigstar, \boxplus, \circledcirc, \bigstar, \bigstar, \spadesuit \circledast\}$$

has 10 elements but they are not all distinct.

If we wanted to count all the permutations of the set S, the number 10! would be too high as some of the elements are indistinguishable and we would be double counting.

For example, using a tree draw list all possible permutations of the set $\{1\spadesuit, 2\spadesuit, 1\clubsuit\}$

Assume that we are only interested in suits of the elements above. Disregard the numbers, and list the permutations above with only their suits.

If we were only interested in calculating the permutations of the set $\{\spadesuit, \spadesuit, \clubsuit\}$ and had used 3! to count the number of permutations.... we would be double counting $(\clubsuit, \spadesuit, \spadesuit)$, $(\spadesuit, \spadesuit, \clubsuit)$ and $(\spadesuit, \clubsuit, \spadesuit)$.

Definition: Assume that a set S has

n_1 elements of type 1
n_2 elements of type 2
.
.
.
n_r elements of type r.

The number of *distinguishable permutations* of the set S is given by

$$\frac{n!}{n_1!n_2!\dots n_r!}.$$

Explanation: Let S be a set with 6 numbers, 3 of which are the number 8. Consider the following permutation

$$(*, 8, 8, *, 8, *).$$

Assume that I can distinguish the three 8's in the set S. Then if I was to count all the permutations with 6! I would be counting the following permutations.

$$\begin{array}{ccc} (*, 8_1, 8_2, *, 8_3, *) & (*, 8_1, 8_3, *, 8_2, *) & (*, 8_2, 8_1, *, 8_3, *) \\ (*, 8_2, 8_3, *, 8_1, *) & (*, 8_3, 8_1, *, 8_2, *) & (*, 8_3, 8_2, *, 8_1, *) \end{array}$$

Note that there are 3! ways that you can order $\{8_1, 8_2, 8_3\}$ which is exactly why we have counted the element $(*, 8, 8, *, 8, *)$ 3! times instead of once. To get rid of rid of this discrepancy we just need to divide 6! by 3!.

Questions

(a) How many ways can we arrange the letters in the word `Mississippi`?

(b) How many ways can I make a necklace of 10 beads of I have 3 red beads, 4 blue beads and 3 white beads?

(c) Assume that I have two urns and balls numbered from 1 to 10. How many ways can I pick out three balls if the order does matter?

(d) Assume that I have two urns and balls numbered from 1 to 10. How many ways can I put three balls into one urn and the seven remaining balls into the other urn?

(e) Assume that I have two urns and n distinguishable balls. How many ways can I pick out k elements if order does matter?

(f) Assume that I have two urns and n distinguishable balls. How many ways can I put k balls into one urn and the remaining $n-k$ balls into the other urn?

Note that in question (d) above the process of putting k elements into an urn and the remaining $n-k$ elements into the other urn is exactly the same thing as picking out a set of k elements from a total set of n elements where *the order does not matter.*

Definition: In a set of n elements a *combination* of k elements is just a subset of the original n-set with k elements with no particular order. The number of k element combinations from a set of n elements is given by $\binom{n}{k}$ where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Questions:

(a) Assume that an urn has ten balls all numbered 1 through 10. If I pull out three balls at once how many possible combinations can I get?

(b) A manager must select 4 employees for a promotion and only 12 are eligible. How many ways can he select the lucky employees?

(c) A class of 10 people must choose a committee of 3 people from the class. How many possible committees can there be?

(d) In a baseball conference with 8 teams, how many games must be played so that each team plays every other team exactly once?

The main difference between permutations and combinations:

Permutations	**Combinations**
choosing k elements from n things where order DOES matter	choosing k elements from n things where order DOES NOT matter
$P(n,k) = \frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$