Probability Distributions and Expected Values

In this section we develop tools to be able to answer the following questions.

(1) In a certain country males are born with .51 probability and females are born with .49 probability. On average, how many males will there be in the country?

(2) Consider the following game. You roll a fair dice and I will pay you $1 for every dot that appears on the top face of the dice after the roll. On average, what is your payoff after playing this game?

(3) I go to Las Vegas to play roulette. On average how much money do I expect to win or lose?

Note that the questions above are interested in the average behavior of random events; the random events in question are

(1) The birth of male.

(2) The value of the top face of a dice after rolling it.

(3) The outcome in a game of roulette.

To understand the outcomes of random events, we must first formalize what a random event is.

Definition: A *random variable* that represents an experiment is a variable that assigns a number to all the possible outcomes of the experiment.

Examples: Write down all the possible values of the following random variables.

(1) X represents tossing a fair dice.

(2) Y represents tossing a coin.

(3) Z represents the number of heads that appear when tossing two fair coins.

(4) $\widetilde{X}$ represents picking a ball from an urn with 1 red ball, 2 blue balls and 3 white balls.

(5) $\widetilde{Y}$ represents if one picks a red ball from an urn with 1 red ball, 2 blue balls and 3 white balls.

Now that we know how to express the outcomes of a random variable, we are interested in finding the probability that a random variable takes a certain value.

Examples: Write down the probabilities for the outcomes that the following random variables can take. Also draw a histogram representing the probabilities that you calculate

(1) X represents tossing a fair dice.

(2) Y represents picking a ball from an urn with 1 red ball, 2 blue balls and 3 white balls.

(3) Z represents the number of heads found when tossing two coins.

Note that each table in the previous page represents a function that assigns a unique value to an outcome of a specified random variable. These functions are called *probability distribution functions* or just simply *probability distributions*.

The probability distribution functions associated to the tables in the previous example are the following.

(1) $f(x) = \begin{cases} 1/6 & \text{if } x = 1, 2, \ldots, 6 \\ 0 & \text{otherwise} \end{cases}$

(2) $f(y) = \begin{cases} 1/6 & \text{if } y = 0 \\ 2/6 & \text{if } y = 1 \\ 3/6 & \text{if } y = 2 \\ 0 & \text{otherwise} \end{cases}$

(3) $f(z) = \begin{cases} 1/4 & \text{if } z = 0 \\ 1/2 & \text{if } z = 1 \\ 1/4 & \text{if } z = 2 \end{cases}$

Nonetheless, when you are asked to write down the probability distribution function of a random variable it will suffice to write down the table of probabilities associate to the random variable. This is best understood through the following examples.

Examples: Write down the probability distribution function for the following random variables and draw the associated histograms.

(1) X represents tossing a fair coin.

(2) Y represents picking a red ball from an urn with 1 red ball, 2 blue balls and 3 white balls.

(3) Z represents the number of heads that appear when tossing four fair coins.

(4) A electronics factory produces televisions, radios, DVD players and computers. They produce a television 70% of the time, a radio 20% of the time, a DVD player 5% and a computer 5% of the time. Set to be X an item produced by the electronics factory.

To compute the *average value* of a random variable we calculate its expected value defined below.

Definition Suppose a random variable X can take on value x_1 with probability p_1, value x_2 with probability p_2, ... and value x_n with probability p_n. The *expected value* of the random X is

$$E(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n.$$

Examples Calculate the expected value of the following random variables.

(1) X represents tossing a fair coin.

(2) Y represents tossing a fair dice.

(3) Z represents the number of heads that appear when tossing two fair coins.

(4) A stock is priced at \$10 per share. In a day the stock will go up by \$5 with 70% chance and it will go down by \$5 with 30% chance. Let W represent the price of the stock the following day.

Examples Suppose we are considering playing the following games.

(1) A fair dice is rolled. If the roll comes out to be 1,4 or 6 then I pay you the face value of the dice in dollars. If the roll comes out to be 2,3 or 5 then you pay me the face value of the dice in dollars. If X represents YOUR expected payoff from this game, what is $E(X)$?

(2) A fair dice is rolled. If the roll comes out to be 4,5 or 6 then I pay you the face value of the dice in dollars. If the roll comes out 1,2 or 3 the you pay me three times the face value of the dice in dollars. If Y represents YOUR expected payoff from this game, what is $E(Y)$?

(3) A fair dice is rolled. If the roll comes out to be a 3 then I pay you \$5, otherwise you pay me \$1. If Z represents your expected payoff from this game, what is $E(Z)$?

(4) Which of these games described in $(1) - (3)$ would we agree to play? Why?

Expected Value For A Binomial Probability
Let X is a random variable representing the number of successes in a binomial experiment where there are n trials and the probability of success is p. Then

$$E(X) = np$$

Questions

(1) **(Problem 23, p.442)** From a group of 2 women and 5 men, a delegation of 2 is selected. Find the expected number of women in the delegation.

(2) **(Problem 24, p.442)** In a club with 20 seniors and 10 junior members, what is the expected number of junior members on a 3-member committee?

(3) **(Problem 25, p.442)** If 2 cards are drawn at one time from a deck of 52 cards, what is the expected number of diamonds?