Sets

We will begin this section with the following example.

Example We have acumulated the following information about some companies.

Company	**Product**	**Location**	**Franchise**
B&T Inc.	Groceries Pet foods Pharmacy	NY	Yes
PetChow Inc.	Pet foods	NJ	No
Gamco	Pharmacy Pet foods	TX	Yes
TAD Inc.	Pharmacy	FL	No
GreenCorp	Groceries Pharmacy	MA	Yes

In order to analyze this information we need to group the companies together depending on their characteristics.

Question: Which companies sell pet food?

Question: Which companies do not have franchises?

Question: Which companies are from the northeast?

Question: Which companies are from the northeast and produce pet food?

We have just grouped these companies together in *sets*. Also, in the last question we have combined the information from two previous sets to form a new set.

These notes are dedicated to mathematicaly formalize the process that we have just done.

Definition A *set* is a collection of objects in which it is possible to determined if a given object is included in the collection. A set S with elements a, b and c is denoted by

$$S = \{a, b, c\}.$$

Sets do not have any order associated to them, so two sets are equal if and only if they containg the same elements. For example, the following sets are equal

$$\{\text{apples, oranges, pears}\} = \{\text{oranges, apples, pears}\} = \{\text{pears, apples, oranges}\}$$

and

$$\{3, 2, 6\} = \{2, 6, 3\} = \{6, 3, 2\}.$$

Exercise Are the following statments true or false?

(1) {3,2,6}={6,2,13}

(2) {24,75,9}={9,75,24}

(3) {3,1,4,1}={4,1,3}

Definition The objects inside a set are called *elements* or *members* of a set. The symbols $\in$ and $\notin$ are used to denote whether an element is *in* or *not in* the set respectively. For example, 5 is in the set $S = \{3, 6, 5\}$ so we write

$$5 \in \{3, 6, 5\} \quad \text{or} \quad 5 \in S.$$

In the case that 4 is not in the set $S = \{3, 6, 5\}$ we write

$$4 \notin \{3, 5, 6\} \quad \text{or} \quad 4 \notin S.$$

Example Are the following statements true or false?

(1) grapes $\notin$ {oranges, apples, pears}

(2) $4 \notin \{2^0, 2^1, 2^2, 2^3\}$

(3) $6 \in \{5, a, \text{tree}, 8, b, 6\}$

Definition If a set has no elemets, we call it the *empty set* and denote it by $\emptyset$.

Definition Let A be a set and denote by

$$n(A) = \text{ the number of elements in set } A$$

BE CAREFUL! The symbols $0, \emptyset, \{0\}, \{\emptyset\}$ all mean different things. See the list below.

(1) $0 =$ the number 0

(2) $\emptyset =$ the set with no elements, $n(\emptyset) = 0$

(3) $\{0\} =$ the set made up of the single element 0, $n(\{0\}) = 1$

(4) $\{\emptyset\} =$ the set made up of the single element $\emptyset$, $n(\{\emptyset\}) = 1$

Exercise Calculate the following.

(1) Let $A = \{3, \emptyset, 0, e, \text{pears}\}$. Then $n(A) =$

(2) Let $B = \{3, 4, 10, 3, 0\}$. Then $n(B) =$

(3) Let $C = \emptyset$. Then $n(C) =$

Sometimes sets have only elements that share a common characteristic. We wante to be able to describe a set using this characteristic without having to write down each element. We do this in the following way

$$\{\text{all possible elements}|\text{chracteristic}\}.$$

To clarify this, lets look at some examples.

(1) The set A contains all even numbers.

$$A = \{x \in \mathbb{N} \mid x = 2k \text{ for some } k \in \mathbb{N}\}$$

Recall that $\mathbb{N} = \{\text{all the natural numbers}\}$.

(2) The set C contains all possitive numbers divisible by 3 that are less than 13.

$$C = \left\{0 < x < 13 \mid \frac{x}{3} \in \mathbb{N}\right\}$$

In this case the set is small enough that we can write down all the elements

$$C = \{3, 6, 9, 12\}$$

but if we just write down all the elements we might miss the characteristic that they all share.

(3) The set B contains all the fruits that are red.

$$B = \{\text{fruits}|\text{the color of the fruit is red}\}$$

Exercise Draw a line matching all the sets.

Natural numbers divisilbe by 3.	$\{x \in \mathbb{N} \mid x < 5\}$
Real numbers less than 5	{Fathers | The person was a US president}
Real numbers between 5 and 100.	$\{x \in \mathbb{R} \mid x < 5\}$
US presidents that were fathers.	$\{x \in \mathbb{N} \mid x = 3k \text{ for some} k \in \mathbb{N}\}$
Natural numbers less than 5.	$\{a \leq x \leq b \mid a = 5 \text{ and } b = 100\}$

Sometimes the elements of one set will all be elements of another set. For example, all the elements of A are elements of B if

$$A = \{3, 6, e\} \text{ and } B = \{e, 4, 6, c, a, 3\}.$$

Definitions: Let A and B be sets.

(1) If all the elements of A are in B we say that A is a *subset* of B and denote it by

$$A \subseteq B.$$

(2) If $A \subset B$ but we know that $A \neq B$ then we say that A is a *proper subset* of B and denote it by

$$A \subset B.$$

Remark: For any set A

$$\emptyset \subseteq A \text{ and } A \subseteq A.$$

Exercise Are the following statements true or false?

(1) $\emptyset \in \{5\}$

(2) $A \subset A$

(3) $\{1, 2, 3, 5, 5\} \subset \{1, 2, 3, 4, 5\}$

(4) $\emptyset \subset \{5\}$

(5) $\{\text{All even numbers}\} \subset \mathbb{N}$

The *Venn diagram* below illustrates the idea of $A \subseteq B$.

Exercise Write down all the subsets of the following sets using a tree diagram.

(1) $A = \{3, 4\}$

$n(\text{all subsets of } A) =$

(2) $B = \{3, \emptyset, 3\}$

$n(\text{all subsets of } B) =$

(3) $C = \{e, 2, d\}$

$n(\text{all subsets of } C) =$

(4) $D = \emptyset$

$n(\text{all subsets of } D) =$

Question: Let A be a set. Can you write a formula for the number of subsets of A?

Exercise Calculate the number of subsets for the following sets.

(1) $A = \{2, \clubsuit\}$

(2) $\emptyset$

(3) $C = \{4, 5, 2, 5, 4\}$

Set Operations

Given some sets, we can create new sets by using the following *set operations* described below. Let U be a *universal set* that contains sets A and B, ie

$$A \subseteq U \quad B \subseteq U.$$

(1) $A' = \{x \in U \mid x \notin A\}$ is the *complement* of A.

Venn diagram:

(2) $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$ is the *union* of sets A and B.

Venn diagram:

(3) $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$ is the *intersection* of sets A and B.

Venn diagram:

Definition: We say that two sets A and B are *disjoint* if $A \cap B = \emptyset$.
Venn diagram:

Exercise Let U be the universal set. Calculate the following sets.

(1) $(A')' =$

(2) $\emptyset' =$ and $U' =$

(3) $(A \cup B)' =$

(4) $A' \cup B' =$

Exercise: Let $U = \{1, 2, 3, 4, 5, 6, \spadesuit, \bigstar, \pounds, \$\}$ be the universal set with $A = \{3, \spadesuit, \pounds, 2, \bigstar, 3, 6, 2\}$ and $B = \{4, 4, \bigstar, \spadesuit\}$. Construct the following sets.

(1) $A' \cap B =$

(2) $(A \cup B')' =$

(3) $A' \cup B' =$