SOME PROBLEMS FOR MATH 6490 [ANSWERS]

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Throughout we will consider algebraic groups and Lie algebras over a general field k unless otherwise specified.

(1) Fix a matrix $S \in M_n(k)$.

(a) Show that the equation $A^tSA = S$ defines a linear algebraic subgroup G of GL_n defined over k. Describe G when S is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

For any k-algebra R, we need to show that $G(R)$ is a subgroup of $\mathrm{GL}_n(R)$. We clearly have $I \in G(R)$, so it suffices to show that $AB \in G(R)$ for any $A, B \in G(R)$. For $A, B \in G(R)$, we have

$$(AB)^tS(AB) = B^t(A^tSA)B = B^tSB = S,$$

so $AB \in G(R)$. This proves that G is a linear algebraic group.

- When $S = 0$, $A^tSA = S$ always holds so $G = \mathrm{GL}_2$.
- Take $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. With $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, a computation shows that $A^tSA = \det(A) \cdot S$. Therefore, $G = \mathrm{SL}_2$. [Note that this, by definition, is also Sp_2.]
- Take $S = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. With $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, a computation shows that $A^tSA - S = \left(\begin{smallmatrix} ac & ad-1 \\ bc & bd \end{smallmatrix}\right)$. So G is given by the equations $ac = 0$, $ad = 1$, $bc = 0$ and $bd = 0$. Therefore, G is the subgroup given by matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$.
- With $S = I$, we have $A^tA = I$ which is the orthogonal group O_2.

(b) Show that the Lie algebra of G is

$$\mathfrak{g} = \{B \in M_n(k) : B^tS + SB = 0\}.$$

Given $B \in M_n(k)$, we have $I + \varepsilon B$ in $G(k[\varepsilon])$ if and only if

$$(I + \varepsilon B)^tS(I + \varepsilon B) = S.$$

Expanding out and using $\varepsilon^2 = 0$, we obtain the equation $S + \varepsilon(B^tS + SB) = S$ and hence $B^tS + SB = 0$.

(2) Let A be a k-algebra. A derivation of A is a k-linear map $D\colon A \to A$ that satisfies

$$D(ab) = aD(b) + D(a)b$$

for all $a, b \in A$.

(a) Show that the set $\mathrm{Der}_k(A)$ of k-linear derivations of A is a Lie algebra where we use the usual function addition and the pairing

$$[D, D'] = D \circ D' - D' \circ D.$$

(As a prototypical example observe that with $k = \mathbb{R}$, differentiation gives a derivation on the ring $C^\infty(\mathbb{R})$ of smooth functions on $\mathbb{R}$.)

Proof omitted. It is hopefully straightforward (but boring).

(b) Now consider a Lie algebra $\mathfrak{g}$ over k; it is a (non-commutative) k-algebra with multiplication given by the bracket. Show that the map $\operatorname{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ given by $y \mapsto [x,y]$ is a derivation of $\mathfrak{g}$. Show that ad gives a homomorphism of Lie algebras $\mathfrak{g} \to \operatorname{Der}_k(\mathfrak{g})$.

That $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{Der}_k(\mathfrak{g})$ is k-linear is clear since $[\ ,\ ]$ is bilinear. We need only check that ad preserve the Lie brackets. Take any $x, y, z \in \mathfrak{g}$. We have

$$\begin{aligned}[\operatorname{ad}(x), \operatorname{ad}(y)](z) &= (\operatorname{ad}(x) \circ \operatorname{ad}(y) - \operatorname{ad}(y) \circ \operatorname{ad}(x))(z)\\ &= [x,[y,z]] - [y,[x,z]]\\ &= [x,[y,z]] + [y,[z,x]];\end{aligned}$$

by the Jacobi identity this equals $-[z,[x,y]] = [[x,y],z] = \operatorname{ad}([x,y])(z)$. Therefore, $\operatorname{ad}([x,y]) = [\operatorname{ad}(x), \operatorname{ad}(y)]$.

(3) Let $\mathfrak{g}$ be a Lie algebra over k. The center of $\mathfrak{g}$ is $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y] = 0 \text{ for all } y \in \mathfrak{g}\}$.

(a) Show that $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

(b) Prove that $\mathfrak{g}/Z(\mathfrak{g})$ is isomorphic to a subalgebra of some $\mathfrak{gl}_n$.
(Hint: consider the adjoint homomorphism $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \operatorname{End}_k(\mathfrak{g})$.)

Let

$$\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

be the adjoint homomorphism; it is a homomorphism of Lie algebras. For a fixed $x \in \mathfrak{g}$, $\operatorname{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ is the homomorphism $y \mapsto [x,y]$. We thus have $\operatorname{ad}(x) = 0$ if and only if $x \in Z(\mathfrak{g})$. Therefore, $\ker(\operatorname{ad}) = Z(\mathfrak{g})$ and hence $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$. The map ad thus induces an injective homomorphism $\mathfrak{g}/Z(\mathfrak{g}) \hookrightarrow \mathfrak{gl}(\mathfrak{g})$. In particular, $\mathfrak{g}/Z(\mathfrak{g})$ is isomorphic to $\mathfrak{gl}_n(k)$ where n is the dimension of $\mathfrak{g}$ over k.

(c) Find the center of $\mathfrak{gl}_n(k)$.

We may assume that $n \geq 2$ ($\mathfrak{gl}_n(k) = k$ is commutative and is hence its own center). The center of $\mathfrak{gl}_n(k)$ consists of the matrices $B \in M_n(k)$ such that $[B,A] = 0$ for all $A \in M_n(k)$; equivalently, B commutes with all matrices in $M_n(k)$.

Fix a matrix $B \in Z(\mathfrak{gl}_n(k))$. Take any distinct $\alpha, \beta \in \{1, \dots, n\}$ and let $E \in M_n(k)$ be the matrix that satisfies $E_{i,j} = \delta_{i,\alpha}\delta_{j,\beta}$. We have $EB - BE = 0$ by assumption on B, so

$$0 = (EB - BE)_{\alpha,j} = \sum_k E_{\alpha,k}B_{k,j} - \sum_k B_{\alpha,k}E_{k,j} = B_{\beta,j} - B_{\alpha,\alpha}\delta_{j,\beta}.$$

For any $j \in \{1, \dots, n\} - \{\beta\}$, we have $0 = B_{\beta,j} - 0 = B_{\beta,j}$. Therefore, $B_{i,j} = 0$ when $i \neq j$ and hence B is diagonal.
For $j = \beta$, we have $0 = B_{\beta,\beta} - B_{\alpha,\alpha}$. Since $\alpha, \beta \in \{1, \dots, n\}$ were arbitrary and distinct, we find that $B_{1,1} = B_{2,2} = \cdots = B_{n,n}$.
We have shown that B is a scalar matrix. Conversely, every scalar matrix lies in the center of $\mathfrak{gl}_n(k)$. Therefore, the center of $\mathfrak{gl}_n(k)$ is kI.

(d) Find the center of $\mathfrak{sl}_n(k)$. (The answer will depend on n and the characteristic of k.)

The center of $\mathfrak{sl}_1(k) = 0$ is 0, so assume that $n \geq 2$.

The same proof as in the last part shows that every matrix in $Z(\mathfrak{gl}_n(k))$ is scalar; note that the matrices E showing up all had trace 0 since α and β were chosen to be distinct.
Therefore, $Z(\mathfrak{sl}_n(k))$ consists of the scalar matrices in $\mathfrak{sl}_n(k)$. For $a \in k$, we have $\operatorname{tr}(aI_n) = na$. If the characteristic of k does not divide n, then $\operatorname{tr}(aI_n) = 0$ if and only if $a = 0$. If the characteristic of k divides n, then $\operatorname{tr}(aI_n) = na = 0$.
Therefore, $Z(\mathfrak{sl}_n(k))$ equals 0 if the characteristic of k does not divide n and kI otherwise.

(4) Show that $\mathfrak{sl}_2(k)$ is simple if the characteristic of k is not 2.

Define the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$

they form a basis of $\mathfrak{sl}_2(k)$. One can check that

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f;$$

these computations determine the Lie bracket since it is alternating and bilinear.

Suppose that $\mathfrak{g}$ is a non-zero ideal of $\mathfrak{sl}_2(k)$. Choose a non-zero element $x \in \mathfrak{g}$; we have $x = ae + bh + cf$ for unique $a, b, c \in k$. We have

$$[x,e] = b[h,e] + c[f,e] = 2be - ch,$$

and hence $[[x,e],e] = -c[h,e] = -2ce$. Therefore, $-2c \cdot e \in \mathfrak{g}$ since $\mathfrak{g}$ is an ideal.

Suppose that $c \neq 0$. We then have $e \in \mathfrak{g}$ (this uses that $\operatorname{char} k \neq 2$). We have $h \in \mathfrak{g}$ since $[e,f] = h$ and hence $f \in \mathfrak{g}$ since $[h,f] = -2f$. Since $\mathfrak{g}$ contains e, h and f, we deduce that $\mathfrak{g} = \mathfrak{sl}_2(k)$.

Now suppose that $c = 0$, i.e., $x = ae + bh$. We have $[x,f] = ah - 2bf$ in $\mathfrak{g}$ since it is an ideal. If $b \neq 0$, then the above case with x replaced by $[x,f]$ shows that $\mathfrak{g} = \mathfrak{sl}_2(k)$.

It remains to consider the case where $x = ae$ with $a \neq 0$. We have $[x,f] = ah$ in $\mathfrak{g}$ since it is an ideal. The previous case then shows that $\mathfrak{g} = \mathfrak{sl}_2(k)$.

[The Lie algebra $\mathfrak{sl}_2(k)$ is not simple when k has characteristic 2. Indeed in the previous exercise we saw that the center of $\mathfrak{sl}_2(k)$ is a non-trivial ideal.]

(5) Let $\mathfrak{g}$ be a Lie algebra over k. The derived algebra of $\mathfrak{g}$ is the k-subspace of $\mathfrak{g}$ generated by $\{[x,y] : x, y \in \mathfrak{g}\}$; we will denote it by $[\mathfrak{g},\mathfrak{g}]$ or $\mathscr{D}\mathfrak{g}$.

(a) Prove that $[\mathfrak{g},\mathfrak{g}]$ is an ideal of $\mathfrak{g}$ and that the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is commutative.

We first show that $\mathscr{D}\mathfrak{g}$ is an ideal. Since $\mathscr{D}\mathfrak{g}$ is a k-subspace of $\mathfrak{g}$, it suffices to show that $[x,y] \in \mathscr{D}\mathfrak{g}$ for any $x \in \mathfrak{g}$ and $y \in \mathscr{D}\mathfrak{g}$. This is clear; $[x,y] \in \mathscr{D}\mathfrak{g}$ for any $x, y \in \mathfrak{g}$.
For any $x, y \in \mathfrak{g}$, we have $[x + \mathscr{D}\mathfrak{g}, y + \mathscr{D}\mathfrak{g}] = [x,y] + \mathscr{D}\mathfrak{g} = \mathscr{D}\mathfrak{g}$. This shows that $\mathfrak{g}/\mathscr{D}\mathfrak{g}$ is commutative.

(b) Show that $[\mathfrak{g},\mathfrak{g}]$ is the smallest ideal $\mathfrak{a}$ of $\mathfrak{g}$ for which $\mathfrak{g}/\mathfrak{a}$ is commutative.

Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ for which $\mathfrak{g}/\mathfrak{a}$ is commutative. For any $x, y \in \mathfrak{g}$, we have $[x,y] \in \mathfrak{a}$ (otherwise their cosets $\bar{x}$ and $\bar{y}$ in $\mathfrak{g}/\mathfrak{a}$ would satisfy $[\bar{x},\bar{y}] \neq 0$). Since $\mathfrak{a}$ is a vector space over k containing all $[x,y]$ with $x, y \in \mathfrak{g}$, we thus have $\mathfrak{a} \supseteq \mathscr{D}\mathfrak{g}$. This and the previous part shows that $\mathscr{D}\mathfrak{g}$ has the desired property.

(6) Define $\mathscr{D}^0\mathfrak{g} = \mathfrak{g}$ and $\mathscr{D}^n\mathfrak{g} = \mathscr{D}(\mathscr{D}^{n-1}\mathfrak{g})$ for $n \geq 1$ (in particular, $\mathscr{D}^1\mathfrak{g} = \mathscr{D}\mathfrak{g}$). We say that $\mathfrak{g}$ is solvable if $\mathscr{D}^n\mathfrak{g} = 0$ for n sufficiently large.

(a) Let $\mathfrak{b}_n$ be the Lie subalgebra of $\mathfrak{gl}_n(k)$ consisting of upper triangular matrices. Prove that $\mathfrak{b}_n$ is solvable.

Let $\mathfrak{h}_m$ be the Lie subalgebra consisting of $B \in \mathfrak{gl}_n(k)$ such that $B_{i,j} = 0$ if $i < j + m$. In particular, $\mathfrak{h}_0 = \mathfrak{b}_n$.
For $B, C \in \mathfrak{b}_n$, we have $[B, C] = BC - CB \in \mathfrak{h}_1$; equivalently, BC and CB have the same diagonal elements. Therefore, $\mathscr{D}\mathfrak{b}_n = \mathscr{D}\mathfrak{h}_0$ is contained in $\mathfrak{h}_1$.
Now take any $m \geq 1$. For any $B, C \in \mathfrak{h}_m$, one can check that BC and CB belong to $\mathfrak{h}_{m+1}$ and hence $[B, C] \in \mathfrak{h}_{m+1}$. Therefore, $\mathscr{D}\mathfrak{h}_m \subseteq h_{m+1}$.
By induction, we find that $\mathscr{D}^n\mathfrak{b}_n = \mathscr{D}^n\mathfrak{h}_0$ is contained in $\mathfrak{h}_n = 0$. Therefore, $\mathfrak{b}_n$ is solvable.

(b) Give an example of a Lie algebra that is not solvable.

Consider a simple Lie algebra $\mathfrak{g}$; we claim that it is non-solvable. Since $\mathscr{D}\mathfrak{g}$ is an ideal of $\mathfrak{g}$, we have $\mathscr{D}\mathfrak{g} = 0$ or $\mathscr{D}\mathfrak{g} = \mathfrak{g}$. If $\mathscr{D}\mathfrak{g} = 0$, then $\mathfrak{g} \cong \mathfrak{g}/\mathscr{D}\mathfrak{g}$ is commutative which is ruled out in the simple assumption. Therefore, $\mathscr{D}\mathfrak{g} = \mathfrak{g}$ and hence $\mathscr{D}^n\mathfrak{g} = \mathfrak{g}$ for all $n \geq 1$. This proves that $\mathfrak{g}$ is non-solvable.
From an earlier exercise, $\mathfrak{sl}_2(\mathbb{C})$ is simple and hence gives an example.

(7) Show that, up to isomorphism, there is a unique non-commutative Lie algebra $\mathfrak{g}$ of dimension 2 over k. (Such a $\mathfrak{g}$ has a basis $\{x, y\}$ satisfying $[x, y] = x$.)

Since $\mathfrak{g}$ is non-commutative, there are $v, w \in \mathfrak{g}$ such that $[v, w] \neq 0$. The pair $\{v, w\}$ is a basis of the 2-dimensional k-vector space $\mathfrak{g}$ (if v and w are linearly dependent, then we would have $[v, w] = 0$). There are thus $a, b \in k$, not both zero, such that

$$[v, w] = av + bw.$$

Without loss of generality, suppose that $a \neq 0$ (otherwise swap v and w). Using that $[w, w] = 0$, we find that $[v + b/a \cdot w, w] = av + bw$. Define $x = v + b/a \cdot w$ and $y = a^{-1}w$; they are linearly independent. Therefore,

$$[x, y] = [v + b/a \cdot w, a^{-1}w] = a^{-1}[v + b/a \cdot w, w] = a^{-1}(av + bw) = v + b/a \cdot w = x.$$

So $\mathfrak{g}$ has a basis $\{x, y\}$ with $[x, y] = x$ (the Lie bracket is uniquely determined by this information).

(8) Find two linear algebraic groups G_1 and G_2 defined over $\mathbb{R}$ that are not isomorphic and satisfy $\#G_1(\mathbb{C}) = \#G_2(\mathbb{C}) = 3$.
[Hint: construct G_1 and G_2 so that $|G_1(\mathbb{R})| \neq |G_2(\mathbb{R})|$.]

First consider the linear algebraic subgroup G_1 of GL_3 given by

$$G(R) = \{I, A, A^2\} \quad \text{where} \quad A = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right).$$

Note that $A^3 = I$ and $\#G(\mathbb{C}) = 3$.

Now consider the linear algebraic subgroup $G_2 := \mu_3$ of $\mathbb{G}_m$ given by the equation $a^3 = 1$, i.e., $G_2(R) = \{a \in R^\times : a^3 = 1\}$. The group $G_2(\mathbb{C})$ consists of the cube roots of unity in $\mathbb{C}$ and hence has order 3.

The groups G_1 and G_2 are not isomorphic (over $\mathbb{R}$) since $\#G_1(\mathbb{R}) = 3$ and $\#G_2(\mathbb{R}) = 1$.

(9) Prove that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not isomorphic linear algebraic groups over k.

It suffices to note that the groups $\mathbb{G}_a(\bar{k}) = \bar{k}$ and $\mathbb{G}_m(\bar{k}) = \bar{k}^\times$ are not isomorphic. If k has characteristic 0, then $\bar{k}$ has no points of finite order except 0 while -1 has order 2 in $\bar{k}^\times$. If k has characteristic $p > 0$, then every element of $\bar{k}$ has order 1 or p, while $\bar{k}^\times$ has no elements of order p.

Alternatively, one could just note that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not even isomorphic as varieties over k; it suffices to prove that the k-algebras $k[x]$ and $k[x, x^{-1}]$ are not isomorphic.

(10) Suppose that k has characteristic $p > 0$. For each k-algebra R, define $\alpha_p(R) = \{a \in R : a^p = 0\}$; it is a subgroup of $\mathbb{G}_a(R) = R$. Show that α_p is an algebraic subgroup of $\mathbb{G}_a$. What is the group $\alpha_p(\bar{k})$? Compute the Lie algebra of α_p.

Let R be any k-algebra. We need to show that $\alpha_p(R)$ is a subgroup of $\mathbb{G}_a(R) = R$. This is clear since $(a + p)^p = a^p + b^p$ for $a, b \in R$ (this uses that $p \cdot 1 = 0$ in R).

If $a \in \bar{k}$ satisfies $a^p = 0$, then $a_p = 0$ since $\bar{k}$ is a field. Therefore, $\alpha_p(\bar{k}) = \{0\}$.

Finally, we compute the Lie algebra $\mathfrak{g}$ of α_p. The Lie algebra is the kernel of $\alpha_p(k[\varepsilon]) \to \alpha_p(k)$ (where as usual we are setting $\varepsilon = 0$). However, $\alpha_p(k) = 0$ so $\mathfrak{g} = \alpha_p(k[\varepsilon])$. For $a + b\varepsilon \in k[\varepsilon]$, we have $(a + b\varepsilon)^p = a^p + b^p \varepsilon^p = a^p$. Therefore, $\mathfrak{g} = \{b\varepsilon : b \in k\}$. So $\mathfrak{g} \cong k$ (and the Lie bracket is trivial). [If you are used to matrix groups, you can view $\mathbb{G}_a$ as the subgroup of GL_2 consisting of upper triangular matrices with 1 on the diagonal terms.]

[We have $\dim_k \mathfrak{g} = 1 > 0 = \dim \alpha_p$ and so the linear algebraic group α_p is not smooth (this cannot happen in characteristic 0).]

(11) If k has characteristic 0, then show that $\mathrm{End}(\mathbb{G}_a) \cong k$, where $\mathbb{G}_a$ is the additive group defined over k. Is this true when k has characteristic $p > 0$?

The coordinate ring of $\mathbb{G}_a$ is $A = k[x]$.

Take any morphism $\mathbb{G}_a \to \mathbb{G}_a$ as a variety over k; it corresponds with a k-algebra homomorphism $\varphi : A \to A$. The homomorphism φ is uniquely determined by the polynomial $f := \varphi(x)$. The morphism $\mathbb{G}_a \to \mathbb{G}_a$ is given by

$$\mathbb{G}_a(R) \to \mathbb{G}_a(R), \quad a \mapsto f(a)$$

for all k-algebras R.

For such a morphism to be a homomorphism of groups, we need to have $f(a + b) = f(a) + f(b)$ for all $a, b \in R$ and all k-algebras R. We thus need the equality $f(x + y) = f(x) + f(y)$ as polynomials in $k[x, y]$.

Let cx^d be the leading term of $f(x)$. Comparing the degree d terms of $f(x + y)$ and $f(x) + f(y)$, we find that

$$c \sum_{i=0}^{d} \binom{d}{i} x^i y^{d-i} = c(x^d + y^d).$$

We have $\operatorname{char} k = 0$, so the only way this can hold is for d to be 0 or 1. We must have $f(0) = 0$ (this follows from $f(a + b) = f(a) + f(b)$), so $f(x) = 0$ or $f(x) = cx$ with $c \in k^\times$.

Conversely, each morphism $f_c : \mathbb{G}_a \to \mathbb{G}_a$, $a \mapsto c \cdot a$ is a homomorphism of algebraic groups. Therefore, the map $k \to \mathrm{End}(\mathbb{G}_a)$, $c \mapsto f_c$ is an isomorphism of groups.

When $\operatorname{char} k = p > 0$, then the morphism $\tau\colon \mathbb{G}_a \to \mathbb{G}_a$, $a \mapsto a^p$ is a homomorphism of algebraic groups that is clearly not of the form $a \mapsto ca$. The kernel of τ is the algebraic group α_p of the previous section.

One can show that the ring $\operatorname{End}(\mathbb{G}_a)$ of automorphism of the linear algebraic group $\mathbb{G}_a$ (defined over k) is isomorphic to the "twisted polynomial ring" $k[\tau]$ (the addition law on $\operatorname{End}(\mathbb{G}_a)$ comes the addition of $\mathbb{G}_a$ and multiplication is composition of functions). The ring $k[\tau]$ consists of polynomials in τ with coefficients in k and satisfies the commutation law $\tau \cdot a = a^p \tau$ (in particular, it is non-commutative if $k \neq \mathbb{F}_p$).

(12) Show that $\operatorname{End}(\mathbb{G}_m) \cong \mathbb{Z}$, where $\mathbb{G}_m$ is the multiplicative group defined over k. More explicitly, show that the only homomorphisms $\mathbb{G}_m \to \mathbb{G}_m$ are of the form $g \mapsto g^n$ for some integer n.

The coordinate ring of $\mathbb{G}_m$ is $A := k[x, x^{-1}]$. The morphisms of $\mathbb{G}_m$ (as a variety) correspond with the k-algebra homomorphisms $\varphi\colon A \to A$. Note that such homomorphisms φ are determined by $\varphi(x)$ and $\varphi(x) \in A$ must be invertible.

Observe that $A^\times = \{cx^n : c \in k^\times, n \in \mathbb{Z}\}$. So every morphism $\mathbb{G}_m \to G_m$ is given by $g \mapsto cg^n$ for some $c \in k^\times$ and $n \in \mathbb{Z}$ (so $\mathbb{G}_m(R) = R^\times \to R^\times = \mathbb{G}_m(R)$ is given by $g \mapsto cg^n$ for each k-algebra R). For a homomorphism of algebraic groups we need 1 to map to 1, so c must be 1.

So a homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ must be of the form $g \to g^n$ for some $n \in \mathbb{Z}$, and it is easy to see that each such map is a homomorphism. Therefore, we have an isomorphism $\mathbb{Z} \xrightarrow{\sim} \operatorname{End}(\mathbb{G}_m)$ given by $n \mapsto (g \mapsto g^n)$.

(13) Observe that $\Gamma = \{(a, e^a) : a \in \mathbb{R}\}$ is a subgroup of $\mathbb{G}_a(\mathbb{R}) \times \mathbb{G}_m(\mathbb{R}) = \mathbb{R} \times \mathbb{R}^\times$. Is Γ the real points of some linear algebraic subgroup of $\mathbb{G}_a \times \mathbb{G}_m$ defined over $\mathbb{R}$?

Suppose that there is a linear algebraic subgroup G of $\mathbb{G}_a \times \mathbb{G}_m$ defined over $\mathbb{R}$ with $G(\mathbb{R}) = \Gamma$. We have $G \neq \mathbb{G}_a \times \mathbb{G}_m$ since otherwise $G(\mathbb{R}) = \mathbb{R} \times \mathbb{R}^\times$. Therefore, there is a non-zero polynomial $f \in k[x, y]$ such that $f(a, e^a) = 0$ for all $a \in \mathbb{R}$.

Let j be the degree of f as a polynomial in y. Let $i \geq 0$ be the largest value for which f has a non-zero term of the form $c \cdot x^i y^j$. By our choice of i and j (and the fact that e^a grows exponentially), we find that $f(a, e^a) \sim ca^i (e^a)^j$ as $a \to +\infty$. In particular, $f(a, e^a) \to +\infty$ or $-\infty$ as $a \to +\infty$. This contradicts that $f(a, e^a) = 0$ for all $a \in \mathbb{R}$.

[*Further remarks* (if you know about real Lie groups): Define the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The k-vector space $\mathfrak{g} = k \cdot A$ is a Lie subalgebra of $\mathfrak{gl}_3(\mathbb{R})$. We have a homomorphism of real Lie groups

$$\exp\colon \mathfrak{g} \to \mathrm{GL}_3(\mathbb{R}), \quad B \mapsto \sum_{n \geq 0} \frac{B^n}{n!}.$$

By the way, this map is not algebraic! The image of exp is the real Lie group consisting of matrices of the form $\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^a \end{pmatrix}$. Identifying $\mathbb{G}_a \times \mathbb{G}_m$ with a closed subgroup of GL_3 in the apparent way, we find that Γ agrees with $\exp(\mathfrak{g})$. In particular, $\mathfrak{g}$ is the Lie algebra of Γ.

Using the above, one can argue that the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}_3(\mathbb{R})$ does not arise as the Lie algebra of a linear *algebraic* subgroup of GL_3.]

(14) Let G be a linear algebraic group defined over k. Let H be a normal closed subgroup of G. Show that $\mathfrak{h} = \mathrm{Lie}(H)$ is an ideal in $\mathfrak{g} = \mathrm{Lie}(G)$.

The group $H(k[\varepsilon])$ is a normal subgroup of $G(k[\varepsilon])$. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is also a normal subgroup of $G(k[\varepsilon])$; it is the kernel of the homomorphism $G(k[\varepsilon]) \to G(k)$ obtained by setting $\varepsilon = 0$. The intersection $H(k[\varepsilon]) \cap \mathfrak{g}$ is thus normal in $\mathfrak{g}$; we are done since this group equals $\mathfrak{h}$ (i.e., the kernel of the homomorphism $H(k[\varepsilon]) \to H(k)$ given by $\varepsilon = 0$).

(15) For simplicity suppose that $k = \mathbb{C}$. Let G be the linear algebraic subgroup of GL_n consisting of monomial matrices (i.e., matrices that have exactly one non-zero entry in each row and each column). What is G°? What is the finite group G/G°?

Let D be the algebraic subgroup of GL_n consisting of diagonal matrices; it is isomorphic to $\mathbb{G}_m^n$ and is connected. The group D is contained in G (invertible diagonal matrices are clearly monomial).

Let H be the (finite) subgroup of GL_n consisting of permutation matrices, i.e., monomial matrices all of whose entries are 0 or 1. Observe that we have a disjoint union

$$G = \coprod_{h \in H} hD.$$

So $G^0 = D$ and the quotient map $H \to G/D$ is an isomorphism.

Observe that the group H is isomorphism to the symmetric group on n letters (it permutes the standard basis $e_1, \dots, e_n$ of $\mathbb{C}^n$).

[Using notation from later in the course: The group D of diagonal matrices is a *maximal torus* of GL_n. The group G is the *normalizer* of D in GL_n. The finite group G/D is the *Weyl group* of GL_n.]

(16) Consider $\mathfrak{g} = \mathbb{R}^3$ with the bilinear pairing $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto x \times y$, where we are taking the cross product. Show that $\mathfrak{g}$, with the pairing, is a Lie algebra isomorphic to $\mathfrak{so}_3(\mathbb{R})$.

The vector space $\mathbb{R}^3$ has a basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. We have

$$e_1 \times e_2 = e_3, \quad e_3 \times e_1 = e_2, \quad e_2 \times e_3 = e_1.$$

The Lie algebra $\mathfrak{so}_3(\mathbb{R})$ consists of skew-symmetric matrices in $M_3(\mathbb{R})$. The matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

form a basis of $\mathfrak{so}_3(\mathbb{R})$. One can check that

$$[A_1, A_2] = A_3, \quad [A_3, A_1] = A_2, \quad [A_2, A_3] = A_1.$$

Let $f : \mathbb{R}^3 \to \mathfrak{so}_3(\mathbb{R})$ be the isomorphism of $\mathbb{R}$-vector spaces for which $f(e_i) = A_i$. The cross product on $\mathfrak{g}$ and Lie bracket on $\mathfrak{so}_3(\mathbb{R})$ are alternating so the above computations show that $f(e_i \times e_j) = [f(e_i), f(e_j)]$. Since both of the pairings are bilinear, we have $f(x \times y) = [f(x), f(y)]$ for all $x, y \in \mathfrak{g}$. This proves that $(\mathbb{R}^3, \times)$ is isomorphic to the Lie algebra $(\mathfrak{so}_3(\mathbb{R}), [\,,\,])$; in particular $(\mathbb{R}^3, \times)$ is a Lie algebra.

(17)

(a) Show that $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}_3(\mathbb{R})$ are not isomorphic. (Hint: One possible approach is to show that $\mathfrak{sl}_2(\mathbb{R})$ has a Lie subalgebra of dimension 2 while $\mathfrak{so}_3(\mathbb{R})$ does not.)

Let $\mathfrak{h}$ be the set of matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & -a \end{smallmatrix}\right)$ with $a, b \in \mathbb{R}$. One can verify that $\mathfrak{h}$ is a 2-dimensional subspace of $\mathfrak{sl}_2(\mathbb{R})$ (it is the Lie algebra of the subgroup of SL_2 consisting of upper triangular matrices).

Suppose that $\mathfrak{so}_3(\mathbb{R})$ has a 2-dimensional Lie subalgebra. By the previous exercise, this implies that $(\mathbb{R}^3, \times)$ has a 2-dimensional Lie subalgebra. There are thus linearly independent vectors $\{v, w\}$ of $\mathbb{R}^3$ such $v \times w = av + bw$ for some $a, b \in k$. However, this is impossible since one knows that $v \times w$ is a non-zero vector that is orthogonal to both v and w.

(b) Show that $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{so}_3(\mathbb{C})$ are isomorphic.

One could play around with matrices and construct an explicit isomorphism. Instead I will give a more involved proof (which though unmotivated, gives a more satisfactory explanation.)

Recall that $V := \mathfrak{sl}_2(\mathbb{C})$ is the vector space over $\mathbb{C}$ of dimension 3 consisting of matrices in $M_2(\mathbb{C})$ with trace 0. We define a symmetric and bilinear pairing

$$\langle\, , \rangle \colon V \times V \to \mathbb{C}, \quad (A, B) \mapsto \mathrm{tr}(AB).$$

Let $\mathrm{O}_V(\mathbb{C})$ be the group of $\mathbb{C}$-linear automorphism that preserve the pairing $\langle\, , \rangle$. Let $\mathrm{SO}_V(\mathbb{C})$ be the subgroup of $\mathrm{O}_V(\mathbb{C})$ consisting of matrices with determinant 1.

We claim that $\mathrm{O}_V(\mathbb{C})$ is isomorphic to $\mathrm{SO}_3(\mathbb{C})$. It suffices to observe that the pairing $\langle\, , \rangle$ is symmetric and non-degenerate, so there is an isomorphism $V \xrightarrow{\sim} \mathbb{C}^3$ of $\mathbb{C}$-vector spaces such that the pairing matches the pairing $(a, b) \mapsto a_1 b_1 + a_2 b_2 + a_3 b_3 = a^t \cdot I \cdot b$ on $\mathbb{C}^3$. (Equivalently, one need only know that for any symmetric matrix $B \in M_3(\mathbb{C})$, there is a matrix Q such that QBQ^t is diagonal with entries 0 and 1; a non-degenerate assumption on B implies that $QBQ^t = I$.)

The group $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation on $V = \mathfrak{sl}_2(\mathbb{C})$ and this action preserves the pairing; this gives a homomorphism

$$\varphi\colon\ \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}_V(\mathbb{C}).$$

The kernel of φ is $\{\pm I\}$.
We actually have $\varphi(\mathrm{SL}_2(\mathbb{C})) \subseteq \mathrm{SO}_V(\mathbb{C})$; this can be checked directly or by using continuity. We thus have a homomorphism

$$\varphi\colon\ \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_V(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$$

whose kernel is $\{\pm I\}$. By dimension and connectedness considerations, one can show that φ is surjective.
Of course the homomorphism arises from a map $\mathrm{SL}_2 \to \mathrm{SO}_3$ of linear algebraic groups defined over $\mathbb{C}$. Taking Lie algebras we obtain a homomorphism

$$d\varphi \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{so}_3(\mathbb{C}).$$

It is an isomorphism of Lie algebras since φ is surjective with finite kernel. (Alternatively, one need only show that $d\varphi$ is non-trivial since $\mathfrak{sl}_2(\mathbb{C})$ is simple and $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{so}_3(\mathbb{C})$ have the same dimension.)