

MODULAR CURVES OF PRIME-POWER LEVEL WITH INFINITELY MANY RATIONAL POINTS

ANDREW V. SUTHERLAND AND DAVID ZYWINA

ABSTRACT. For each open subgroup G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ containing $-I$ with full determinant, let X_G/\mathbb{Q} denote the modular curve that loosely parametrizes elliptic curves whose Galois representation, which arises from the Galois action on its torsion points, has image contained in G . Up to conjugacy, we determine a complete list of the 248 such groups G of prime power level for which $X_G(\mathbb{Q})$ is infinite. For each G , we also construct explicit maps from each X_G to the j -line. This list consists of 220 modular curves of genus 0 and 28 modular curves of genus 1. For each prime ℓ , these results provide an explicit classification of the possible images of ℓ -adic Galois representations arising from elliptic curves over \mathbb{Q} that is complete except for a finite set of exceptional j -invariants.

1. INTRODUCTION

Let E be an elliptic curve defined over \mathbb{Q} and denote its j -invariant by j_E . For each positive integer N , let $E[N]$ denote the N -torsion subgroup of $E(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is a fixed algebraic closure of \mathbb{Q} . The natural action of the absolute Galois group $\mathrm{Gal}_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$ induces a Galois representation

$$\rho_{E,N}: \mathrm{Gal}_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

After choosing compatible bases for the torsion subgroups $E[N]$, these representations determine a Galois representation

$$\rho_E: \mathrm{Gal}_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\hat{\mathbb{Z}})$$

whose composition with the projection $\mathrm{GL}_2(\hat{\mathbb{Z}}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ given by reduction modulo N is equal to $\rho_{E,N}$, for each N . The images of $\rho_{E,N}$ and ρ_E are uniquely determined up to conjugacy in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\mathrm{GL}_2(\hat{\mathbb{Z}})$, respectively. If E does not have complex multiplication (CM), then $\rho_E(\mathrm{Gal}_{\mathbb{Q}})$ is an open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$, by Serre's open image theorem [Ser72], hence of finite index in $\mathrm{GL}_2(\hat{\mathbb{Z}})$.¹

Let G be an open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ that satisfies $\det(G) = \hat{\mathbb{Z}}^\times$ and $-I \in G$. Let N be the least positive integer such that G is the inverse image of its image under the reduction map $\mathrm{GL}_2(\hat{\mathbb{Z}}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$; we call N the *level* of G .

Associated to G is a modular curve X_G/\mathbb{Q} ; one can define X_G as the generic fiber of the smooth proper $\mathbb{Z}[1/N]$ -scheme that is the coarse moduli space for the algebraic stack $\mathcal{M}_{\overline{G}}[1/N]$ in the sense of [DR73, §IV], where \overline{G} denotes the image of G under reduction modulo N . See §2 for some background on X_G and an alternate description; in particular, it is a smooth projective geometrically integral curve defined over \mathbb{Q} .

When $G = \mathrm{GL}_2(\hat{\mathbb{Z}})$, the modular curve X_G is the j -line $\mathbb{P}_{\mathbb{Q}}^1 = \mathbb{A}_{\mathbb{Q}}^1 \cup \{\infty\}$. If G and G' are open subgroups of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ with $\det(G) = \det(G') = \hat{\mathbb{Z}}^\times$ and $-I \in G, G'$ such that $G \subseteq G'$, then there

2000 *Mathematics Subject Classification*. Primary 14G35; Secondary 11G05, 11F80.

The first author was supported by NSF grants DMS-1115455 and DMS-1522526.

¹In fact, the open subgroups of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ are exactly those of finite index, by a theorem of Nikolov and Segal [NS03].

is a natural morphism $X_G \rightarrow X_{G'}$ of degree $[G' : G]$. In particular, with $G' = \mathrm{GL}_2(\hat{\mathbb{Z}})$, we have a morphism

$$\pi_G: X_G \rightarrow \mathbb{P}_{\mathbb{Q}}^1 = \mathbb{A}_{\mathbb{Q}}^1 \cup \{\infty\}$$

of degree $[\mathrm{GL}_2(\hat{\mathbb{Z}}) : G]$ from X_G to the j -line.

The key property for our applications is that for an elliptic curve E/\mathbb{Q} with $j_E \notin \{0, 1728\}$, the group $\rho_E(\mathrm{Gal}_{\mathbb{Q}})$ is conjugate in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ to a subgroup of G if and only if j_E is an element of $\pi_G(X_G(\mathbb{Q}))$; see Proposition 2.7. This property requires $-I \in G$, since there is always an elliptic curve E with any given rational j -invariant such that $-I \in \rho_E(\mathrm{Gal}_{\mathbb{Q}})$; it also requires $\det(G) = \hat{\mathbb{Z}}^\times$, since $\det(\rho_E(\mathrm{Gal}_{\mathbb{Q}})) = \hat{\mathbb{Z}}^\times$, and that G contain an element corresponding to complex conjugation.

We are interested in those groups G for which X_G has infinitely many rational points; equivalently, for which there are infinitely many elliptic curves E/\mathbb{Q} , with distinct j -invariants, such that $\rho_E(\mathrm{Gal}_{\mathbb{Q}})$ is conjugate to a subgroup of G . We need only consider modular curves X_G of genus 0 or 1 since otherwise $X_G(\mathbb{Q})$ is finite by Faltings' Theorem [Fal83].

In this article, we give an explicit description of all such subgroups $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ for which the modular curve X_G has infinitely many rational points in the special case where the level N of G is a *prime power*; we also give an explicit model for X_G and the morphism π_G . We need only describe the groups G up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$. For notational simplicity, we define the genus of G to be the genus of the corresponding curve X_G .

Theorem 1.1. *Up to conjugacy, there are 248 open subgroups G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of prime power level satisfying $-I \in G$ and $\det(G) = \hat{\mathbb{Z}}^\times$ for which X_G has infinitely many rational points. Of these 248 groups, there are 220 of genus 0 and 28 of genus 1.*

The 220 subgroups of genus 0 in Theorem 1.1 are given in Tables 1, 2 and 3 in Appendix A. For such a group G of genus 0, we also describe the morphism π_G . More precisely, we give a rational function $J(t) \in \mathbb{Q}(t)$ such that the function field of X_G is of the form $\mathbb{Q}(t)$ and the morphism from X_G to the j -line is given by the equation $j = J(t)$. In particular, if E/\mathbb{Q} is an elliptic curve with $j_E \notin \{0, 1728\}$, then $\rho_E(\mathrm{Gal}_{\mathbb{Q}})$ is conjugate to a subgroup of G if and only if $j_E = J(t_0)$ for some $t_0 \in \mathbb{Q} \cup \{\infty\}$.

The 28 subgroups of genus 1 in Theorem 1.1 are listed in Table 4 of Appendix A; their levels are all powers of 2 except for a group of level 11 whose image in $\mathrm{GL}_2(\mathbb{Z}/11\mathbb{Z})$ is the normalizer of a non-split Cartan subgroup. For such a group G of genus 1, we give a Weierstrass model for X_G and the morphism π_G to the j -line.

Example 1.2. Up to conjugacy, there is a unique subgroup $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ of genus 0 and level 27 given by Theorem 1.1. It has label 27A⁰-27a in our classification, and we may choose it so that the image of G in $\mathrm{GL}_2(\mathbb{Z}/27\mathbb{Z})$ is generated by the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 9 & 5 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. Using Table 1, associated to G is the rational function

$$J(t) = F_3(F_2(F_1(t))) = \frac{(t^3 + 3)^3(t^9 + 9t^6 + 27t^3 + 3)^3}{t^3(t^6 + 9t^3 + 27)},$$

where $F_1(t) = t^3$, $F_2(t) = t(t^2 + 9t + 27)$ and $F_3(t) = (t+3)^3(t+27)/t$. That $J(t)$ is the composition of three rational functions reflects the fact that the morphism π_G factors as $X_G \rightarrow X_{G'} \rightarrow X_{G''} \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ for some groups $G \subsetneq G' \subsetneq G'' \subsetneq \mathrm{GL}_2(\hat{\mathbb{Z}})$. The groups G' and G'' have labels 9B⁰-9a and 3B⁰-3a, respectively, and can also be found in Table 1.

Remark 1.3. In contrast to the case of prime power level, in general there are infinitely many open subgroups G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ satisfying $-I \in G$ and $\det(G) = \hat{\mathbb{Z}}^\times$ for which the modular curve X_G has infinitely many rational points. Let us explicitly construct just one of several infinite families of such groups G .

Let D be the discriminant of a quadratic number field and let $\chi_D: \hat{\mathbb{Z}}^\times \rightarrow \{\pm 1\}$ be the continuous quadratic character arising from the corresponding Dirichlet character. Let $\varepsilon: \mathrm{GL}_2(\hat{\mathbb{Z}}) \rightarrow \{\pm 1\}$ be the character obtained by composing the reduction map $\mathrm{GL}_2(\hat{\mathbb{Z}}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ with the unique non-trivial homomorphism $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \rightarrow \{\pm 1\}$. Define the group

$$G_D := \{A \in \mathrm{GL}_2(\hat{\mathbb{Z}}) : \varepsilon(A) = \chi_D(\det(A))\};$$

it is an open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of index 2 containing $-I$ with $\det(G_D) = \hat{\mathbb{Z}}^\times$ whose level is $|D|$ or $2|D|$, depending on whether $D \equiv 0 \pmod{4}$ or $D \equiv 1 \pmod{4}$. For $D \neq D'$, the groups G_D and $G_{D'}$ are not conjugate in $\mathrm{GL}_2(\hat{\mathbb{Z}})$.

The modular curve X_{G_D} has genus 0 and a rational point (it has a unique, hence rational, cusp); its function field of X_{G_D} is of the form $\mathbb{Q}(t)$ with the map to the j -line given by $J(t) = Dt^2 + 1728$. Each X_{G_D} is a $\mathbb{Q}(\sqrt{D})$ -twist of the modular curve X_G corresponding to the unique index 2 subgroup $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ whose reduction has index 2 in $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$; it has label $2A^0-2a$ in our classification and can be found in Table 3, along with its map to the j -line, which is $J(t) = t^2 + 1728$.

In general, if $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a fixed congruence subgroup of level N and index m containing $-I$, there will be infinitely many non-conjugate open subgroups $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ of index M containing $-I$ with $\det(G) = \hat{\mathbb{Z}}^\times$ whose reductions modulo N coincide with that of Γ upon intersection with $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. The levels M of these groups G may be arbitrarily large multiples of N (and divisible by arbitrarily large primes). The corresponding modular curves X_G/\mathbb{Q} are non-isomorphic, but for each X_G there is a cyclotomic field $\mathbb{Q}(\zeta_M)$ over which X_G becomes isomorphic to the modular curve $X_\Gamma/\mathbb{Q}(\zeta_N)$ (the quotient of the extended upper half plane by the action of Γ) after base change; as in our example, the X_G form an infinite family of twists.

1.1. **ℓ -adic representations.** Fix a prime ℓ . Define the set

$$\mathcal{J}_\ell := \bigcup_G (\pi_G(X_G(\mathbb{Q})) \cap \mathbb{Q})$$

of rational numbers, where G varies over the open subgroups of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ whose level is a power of ℓ and satisfies $-I \in G$ and $\det(G) = \hat{\mathbb{Z}}^\times$, and for which $X_G(\mathbb{Q})$ is finite. Note that the set \mathcal{J}_ℓ contains the 13 j -invariants of CM elliptic curves over \mathbb{Q} : for $n \geq 1$ each CM j -invariant corresponds to points on at least one of the modular curves $X_s^+(\ell^n)$, $X_{ns}^+(\ell^n)$, $X_0(\ell^n)$, and for sufficiently large n these curves have genus at least 2, hence finitely many rational points (by Faltings' Theorem).

For an elliptic curve E/\mathbb{Q} , let

$$\rho_{E,\ell^\infty}: \mathrm{Gal}_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{Z}_\ell)$$

be the representation describing the Galois action on the ℓ -power torsion points; it is the composition of ρ_E with the natural projection $\mathrm{GL}_2(\hat{\mathbb{Z}}) \rightarrow \mathrm{GL}_2(\mathbb{Z}_\ell)$. After excluding a finite number of j -invariants, we will describe the possible images of the ℓ -adic representation arising from elliptic curves over \mathbb{Q} . Denote by $\pm\rho_E(\mathrm{Gal}_{\mathbb{Q}})$ the group generated by $-I$ and $\rho_E(\mathrm{Gal}_{\mathbb{Q}})$.

The following theorem describes the possibilities for $\pm\rho_E(\mathrm{Gal}_{\mathbb{Q}})$, up to conjugacy, when j_E is not in the (finite!) set \mathcal{J}_ℓ .

Theorem 1.4.

- (i) *The set \mathcal{J}_ℓ is finite.*
- (ii) *If E/\mathbb{Q} is an elliptic curve with $j_E \notin \mathcal{J}_\ell$, then $\pm\rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbb{Q}})$ is conjugate in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ to the ℓ -adic projection of a unique group G from Theorem 1.1 with ℓ -power level. Moreover, G does not have genus 1, level 16, and index 24 in $\mathrm{GL}_2(\hat{\mathbb{Z}})$.*
- (iii) *Let G be a group from Theorem 1.1 with ℓ -power level that does not have genus 1, level 16, and index 24 in $\mathrm{GL}_2(\hat{\mathbb{Z}})$. Then there are infinitely many elliptic curves E/\mathbb{Q} , with distinct j -invariants, such that $\pm\rho_E(\mathrm{Gal}_{\mathbb{Q}})$ is conjugate in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ to the ℓ -adic projection of G .*

Remark 1.5.

- (i) Serre has asked whether $\rho_{E,\ell}$ is surjective for all non-CM elliptic curves E/\mathbb{Q} and all primes $\ell > 37$, cf. [Ser81, p. 399]. For $\ell > 37$, this would imply that the set \mathcal{J}_ℓ consists of only the 13 j -invariants of CM elliptic curves over \mathbb{Q} .
- (ii) It would be nice to explicitly know the finite sets \mathcal{J}_ℓ ; the proof that \mathcal{J}_ℓ is finite relies on [Zyw15b], which is ineffective since it applies Faltings' Theorem several times.

Theorem 1.4 describes the subgroups of $\mathrm{GL}_2(\mathbb{Z}_\ell)$, up to conjugacy, that occur as $\pm\rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbb{Q}})$ for infinitely many elliptic curves E/\mathbb{Q} with distinct j -invariants.

Theorem 1.4 also allows us to determine the subgroups of $\mathrm{GL}_2(\mathbb{Z}_\ell)$, up to conjugacy, that occur as $\rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbb{Q}})$ for infinitely many elliptic curves E/\mathbb{Q} with distinct j -invariants. They are precisely the subgroups H of the ℓ -adic projection G of a group from Theorem 1.4 with ℓ -power level such that $\pm H = G$. Indeed if $G = \pm\rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbb{Q}})$, then for any such H there is a quadratic twist of E such that H is conjugate to $\rho_{E',\ell^\infty}(\mathrm{Gal}_{\mathbb{Q}})$, cf. [Zyw15a, §5.1] and [Sut16, §5.6]; when H is properly contained in G this quadratic twist is unique up to isomorphism and can be explicitly determined.

Corollary 1.6. *For $\ell = 2, 3, 5, 7, 11, 13$ there are respectively 1201, 47, 23, 15, 2, 11 subgroups H of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ that arise as $\rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbb{Q}})$ for infinitely many elliptic curves E/\mathbb{Q} with distinct j -invariants; for $\ell > 13$ the only such subgroup is $H = \mathrm{GL}_2(\mathbb{Z}_\ell)$.*

A list of the groups H appearing in Corollary 1.6 can be found in electronic form at [SZ16].

1.2. Overview. We now give a brief overview of the contents of this paper. As already noted, the groups G from Theorem 1.1, along with the corresponding modular curves X_G and morphisms π_G , can be found in Appendix A

In §2, we review the background material we need concerning the modular curves X_G . If G has level N , then we can identify the function field of X_G with a subfield of the field \mathcal{F}_N of modular functions on $\Gamma(N)$ whose Fourier coefficients lie in the cyclotomic field $\mathbb{Q}(\zeta_N)$. As a working definition of X_G , we define it in terms of its function field.

In §3, we determine up to conjugacy the open subgroups G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ with genus at most 1 that satisfy $\det(G) = \hat{\mathbb{Z}}^\times$, $-I \in G$ and contain an element that “looks like complex conjugation”; this last condition is necessary, since otherwise $X_G(\mathbb{R})$, and therefore $X_G(\mathbb{Q})$, is empty. We are left with 220 groups of genus 0 and 250 groups of genus 1 that include all the groups that appear in Theorem 1.1. These computations make use of the tables of Cummins and Pauli of congruence subgroups of low genus [CP03].

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let X_Γ be the smooth compact Riemann surface obtained by taking the quotient of the complex upper-half plane by Γ and adjoining cusps. Assume further that X_Γ has genus 0. In §4, we describe how to explicitly construct a *hauptmodul* for Γ ; it is a meromorphic function h on X_Γ that has a unique pole at the cusp at ∞ . We describe h in terms of *Siegel functions*; its Fourier coefficients are computable and lie in the field $\mathbb{Q}(\zeta_N) \subseteq \mathbb{C}$.

In §5, we prove the part of Theorem 1.1 concerning genus 0 groups. Let G be one of the genus 0 groups from §3 and let $J(t) \in \mathbb{Q}(t)$ be the corresponding rational function from Appendix A. We need to verify that the function field $\mathbb{Q}(X_G)$ of X_G is of the form $\mathbb{Q}(f)$, for some modular function f for which $J(f)$ coincides with the modular j -function. Using our work in §4, we can construct an explicit modular function h such that $\mathbb{Q}(\zeta_N)(X_G) = \mathbb{Q}(\zeta_N)(h)$, along with a rational function $J'(t) \in \mathbb{Q}(\zeta_N)(t)$ such that $J'(h) = j$. The function f must satisfy $f = \psi(h)$ for some $\psi(t) \in \mathbb{Q}(\zeta_N)(t)$ of degree 1, and therefore $J'(h) = j = J(f) = J(\psi(h))$; this in turn implies that $J'(t) = J(\psi(t))$. We then directly test all the modular functions $f := \psi(h)$, where $\psi(t) \in \mathbb{Q}(\zeta_N)(t)$ is one of the finitely many degree 1 rational functions that satisfy $J'(t) = J(\psi(t))$.

In §6, we prove the part of Theorem 1.1 concerning genus 1 groups. Let G be one of the genus 1 groups from §3. One can show that X_G has good reduction at all primes $p \nmid N$ and its modular

interpretation gives a way to compute $\#X_G(\mathbb{F}_p)$ directly from the group G , without requiring an explicit model. By computing $\#X_G(\mathbb{F}_p)$ for enough primes $p \nmid N$, one can determine the Jacobian J_G of X_G up to isogeny. This allows us to compute the rank of $J_G(\mathbb{Q})$ which is an isogeny invariant of J_G . We need only consider groups for which $J_G(\mathbb{Q})$ has positive rank since otherwise $X_G(\mathbb{Q})$ is finite; this leaves the 28 genus 1 groups in Theorem 1.1. These 28 groups G of genus 1 and a description of their morphisms π_G already appear in the literature; our contribution lies in proving that there are no others.

In §7, we complete the proof of Theorem 1.4, and in §8 we explain how we found the rational functions $J(t) \in \mathbb{Q}(t)$ whose verification is described in §5.

Appendix A lists the 248 groups G that appear in Theorem 1.1, along with explicit maps from X_G to the j -line; for the 220 groups of genus 0 these are rational functions $J(t)$, and for the 28 groups of genus 1 these are morphisms $J(x, y)$ from an explicit Weierstrass model for X_G as an elliptic curve of positive rank. One can use these maps to explicitly construct infinite families of elliptic curves E/\mathbb{Q} with distinct j -invariants whose ℓ -adic Galois images match the groups G listed in Theorem 1.4 and the groups H listed in Corollary 1.6 by choosing appropriate quadratic twists.

1.3. Related results. Contemporaneous with our work, Rouse and Zureick-Brown independently computed explicit models for all modular curves X_G/\mathbb{Q} of 2-power level that have a non-cuspidal rational point, including all those for which $X_G(\mathbb{Q})$ is infinite [RZB15]. The X_G of 2-power level in our list agree with theirs, although we generally obtain different (but isomorphic) models (N.B. our groups are transposed relative to theirs; in our choice of the isomorphism $\text{Aut}(E[N]) \simeq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ we view matrices in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ as acting on the left, rather than the right).

Acknowledgements. We thank Jeremy Rouse and David Zureick-Brown for the feedback on an early draft of this article, and the referees for their careful review and helpful comments.

Notation and Terminology. For each integer $n \geq 1$, we denote by ζ_n the n th root of unity $e^{2\pi i/n}$ in \mathbb{C} , and let $K_n := \mathbb{Q}(\zeta_n)$ denote the corresponding cyclotomic field. For any non-constant function $f \in K(t)$, where K is a field, the *degree* of f is its degree as a morphism $\mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$.

For any ring R , we denote by $M_2(R)$ the ring of 2×2 matrices with coefficients in R . We denote by $\hat{\mathbb{Z}}$ the profinite completion of \mathbb{Z} , and view the profinite group

$$\text{GL}_2(\hat{\mathbb{Z}}) \simeq \varprojlim_N \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq \prod_{\ell} \text{GL}_2(\mathbb{Z}_{\ell})$$

as a topological group in the profinite topology. If G is an open subgroup of $\text{GL}_2(\hat{\mathbb{Z}})$, we define its *level* to be the least positive integer N for which G is the inverse image of a subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ under the natural projection $\text{GL}_2(\hat{\mathbb{Z}}) \rightarrow \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. If G is a subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, its level is defined to be the level of its inverse image in $\text{GL}_2(\hat{\mathbb{Z}})$, which is necessarily a divisor of N . For convenience we may identify the level N subgroups of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ with their inverse images in $\text{GL}_2(\hat{\mathbb{Z}})$, and conversely. By the *genus* of an open subgroup G of $\text{GL}_2(\hat{\mathbb{Z}})$ satisfying $-I \in G$ and $\det(G) = \hat{\mathbb{Z}}^{\times}$, we mean the genus of the modular curve X_G defined in §2.

For sets S and T we use $S - T$ to denote the set of elements that lie in S but not T .

2. MODULAR FUNCTIONS AND MODULAR CURVES

In this section, we summarize the background we need concerning modular curves.

2.1. Congruence subgroups. Fix a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, i.e., a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing

$$\Gamma(N) := \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv I \pmod{N}\}$$

for some integer $N \geq 1$. The smallest such N is the *level* of Γ .

The group Γ acts on the complex upper half plane \mathbb{H} by linear fractional transformations, and the quotient $Y_\Gamma = \Gamma \backslash \mathbb{H}$ is a smooth Riemann surface. By adding *cusps*, we can extend Y_Γ to a smooth compact Riemann surface X_Γ . We denote by $X(N)$ the Riemann surface $X_{\Gamma(N)}$. The *genus* of Γ is the genus of the Riemann surface X_Γ .

2.2. Cusps. Define the extended upper half plane by $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. The action of Γ extends to \mathbb{H}^* and we can identify the quotient $\Gamma \backslash \mathbb{H}^*$ with X_Γ . In particular, the cusps correspond to the Γ -orbits of $\mathbb{Q} \cup \{\infty\}$.

Lemma 2.1. *Let a/b and α/β be elements of $\mathbb{Q} \cup \{\infty\}$ with $\gcd(a, b) = \gcd(\alpha, \beta) = 1$ (where we take $\infty = \pm 1/0$). Then $\Gamma \cdot a/b = \Gamma \cdot \alpha/\beta$ if and only if $\gamma \begin{pmatrix} a \\ b \end{pmatrix} \equiv \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \pmod{N}$ for some $\gamma \in \Gamma$.*

Proof. For the case $\Gamma = \Gamma(N)$, see [Shi94, Lemma 1.42]. The general case follows easily. \square

Let $\pm\Gamma$ be the congruence subgroup generated by $-I$ and Γ . From Lemma 2.1, we find that the cusps of X_Γ correspond with the orbits of $\pm\Gamma$ on the set of $\begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^2$ of order N . Using this, it is straightforward to find representatives of the cusps of X_Γ .

2.3. Modular functions. A *modular function* for Γ is a meromorphic function of X_Γ ; they correspond to meromorphic functions f of \mathbb{H} that satisfy $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \Gamma$ and are meromorphic at the cusps. The function field $\mathbb{C}(X_\Gamma)$ of X_Γ consists of the meromorphic functions of X_Γ .

Let τ be a variable of the upper half plane. Let w be the width of the cusp at ∞ , i.e., the smallest positive integer for which $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ is an element of Γ ; it is a divisor of N . For any rational number m , define $q^m := e^{2\pi im\tau}$. Then any modular function f for Γ has a unique q -expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} c_n q^{n/w},$$

where the c_n are complex numbers that are 0 for all but finitely many $n < 0$. We will often refer to the c_n as the *coefficients* of f .

2.4. Field of modular functions. Fix a positive integer N . Denote by \mathcal{F}_N the field of meromorphic functions of the Riemann surface $X(N)$ whose q -expansions have coefficients in $K_N := \mathbb{Q}(\zeta_N)$. For example, $\mathcal{F}_1 = \mathbb{Q}(j)$, where j is the modular j -invariant.

For $f \in \mathcal{F}_N$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, let $f|_\gamma \in \mathcal{F}_N$ denote the modular function satisfying $f|_\gamma(\tau) = f(\gamma\tau)$.

For each $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, let σ_d be the automorphism of K_N satisfying $\sigma_d(\zeta_N) = \zeta_N^d$. We extend σ_d to an automorphism of the field \mathcal{F}_N by defining

$$\sigma_d(f) := \sum_n \sigma_d(c_n) q^{n/N},$$

where f has expansion $\sum_n c_n q^{n/N}$. We now recall some facts about the extension \mathcal{F}_N of $\mathcal{F}_1 = \mathbb{Q}(j)$.

Proposition 2.2. *The extension \mathcal{F}_N of $\mathbb{Q}(j)$ is Galois. There is a unique isomorphism*

$$\theta_N : \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) / \{\pm I\} \xrightarrow{\sim} \mathrm{Gal}(\mathcal{F}_N / \mathbb{Q}(j))$$

such that the following hold for all $f \in \mathcal{F}_N$:

- (a) For $g \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, we have $\theta_N(g)f = f|_{\gamma^t}$, where γ is any matrix in $\mathrm{SL}_2(\mathbb{Z})$ that is congruent to g modulo N and γ^t is the transpose of γ .
- (b) For $g = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we have $\theta_N(g)f = \sigma_d(f)$.

Moreover, the algebraic closure of \mathbb{Q} in \mathcal{F}_N is $\mathbb{Q}(\zeta_N)$; it corresponds to the subgroup $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) / \{\pm I\}$.

Proof. This is well known; see [KL81, Ch. 2 §2] for a summary (where the action given is a right action obtained as above but without the transpose in (a)). \square

Throughout the rest of the paper, we let $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ act on \mathcal{F}_N via the homomorphism θ_N of Proposition 2.2. We set $g_*(f) := \theta_N(g)(f)$ for $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $f \in \mathcal{F}_N$.

Remark 2.3. There are other natural actions of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on \mathcal{F}_N ; for example, one could replace γ^t in condition (a) by γ^{-1} or just act on the right. Our choice is motivated by Proposition 2.6 below.

2.5. Modular curves. Let G be a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ satisfying $-I \in G$ and $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$. Let \mathcal{F}_N^G be the subfield of \mathcal{F}_N fixed by the action of G from Proposition 2.2. Proposition 2.2 and the assumption $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$ imply that \mathbb{Q} is algebraically closed in \mathcal{F}_N^G .

The *modular curve* X_G associated with G is the smooth projective curve with function field \mathcal{F}_N^G . The curve X_G is defined over \mathbb{Q} and is geometrically irreducible. The inclusion of fields $\mathcal{F}_N^G \supseteq \mathcal{F}_1 = \mathbb{Q}(j)$ gives rise to a non-constant morphism

$$\pi_G: X_G \rightarrow \mathrm{Spec} \mathbb{Q}[j] \cup \{\infty\} = \mathbb{P}_{\mathbb{Q}}^1$$

of degree $[\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : G]$. Moreover, given another group $G \subseteq G' \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, the inclusion $\mathcal{F}_N^{G'} \subseteq \mathcal{F}_N^G$ induces a nonconstant morphism $X_G \rightarrow X_{G'}$ of degree $[G' : G]$. Composing $X_G \rightarrow X_{G'}$ with $\pi_{G'}$ gives the morphism π_G .

Let Γ be the congruence subgroup consisting of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ for which γ^t modulo N lies in $G \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. The level of Γ divides, but need not equal, N .

Lemma 2.4.

- (i) *The field $K_N(X_G)$, i.e., the function field of the base extension of X_G to K_N , is the field consisting of $f \in \mathcal{F}_N$ satisfying $f|_\gamma = f$ for all $\gamma \in \Gamma$.*
- (ii) *The genus of the modular curve X_G is equal to the genus of Γ .*

Proof. Proposition 2.2 implies that K_N is algebraically closed in \mathcal{F}_N and that we have an isomorphism $\mathrm{Gal}(\mathcal{F}_N/K_N(j)) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}$. Thus $K_N(X_G)$ is the subfield of \mathcal{F}_N fixed by $G \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Part (i) is now clear.

Since K_N is algebraically closed in \mathcal{F}_N and \mathbb{Q} is algebraically closed in $\mathbb{Q}(X_G)$, we have

$$[\mathbb{C} \cdot K_N(X_G) : \mathbb{C}(j)] = [K_N(X_G) : K_N(j)] = [\mathbb{Q}(X_G) : \mathbb{Q}(j)] = [\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : G].$$

Since $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$, we deduce that $[\mathbb{C} \cdot K_N(X_G) : \mathbb{C}(j)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$.

Clearly each $f \in K_N(X_G)$ is a modular function for Γ , thus $\mathbb{C} \cdot K_N(X_G) \subseteq \mathbb{C}(X_\Gamma)$. We in fact have $\mathbb{C} \cdot K_N(X_G) = \mathbb{C}(X_\Gamma)$, since $[\mathbb{C} \cdot K_N(X_G) : \mathbb{C}(j)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = [\mathbb{C}(X_\Gamma) : \mathbb{C}(j)]$. The curve X_G has the same genus as the Riemann surface X_Γ because $\mathbb{C}(X_G) = \mathbb{C}(X_\Gamma)$. \square

Remark 2.5. Another natural congruence subgroup to study is the the congruence subgroup Γ' consisting of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that γ modulo N lies in $G \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, which we use later in the paper. Observe that the congruence subgroups Γ and Γ' are conjugate in $\mathrm{SL}_2(\mathbb{Z})$; indeed, we have $B^{-1}\gamma B = (\gamma^t)^{-1}$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where $B := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus Γ and Γ' have the same genus.

The following proposition is crucial to our application.

Proposition 2.6. *Let E be an elliptic curve defined over \mathbb{Q} with $j_E \notin \{0, 1728\}$. Then $\rho_{E,N}(\mathrm{Gal}_{\mathbb{Q}})$ is conjugate in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to a subgroup of G if and only if j_E belongs to $\pi_G(X_G(\mathbb{Q}))$.*

Proof. See §3 of [Zyw15a] for a proof. \square

2.6. Modular curves and open subgroups. Fix an open subgroup G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ that satisfies $-I \in G$ and $\det(G) = \hat{\mathbb{Z}}^\times$. Let $N \geq 1$ be an integer that is divisible by the level of G . Define the modular curve

$$X_G := X_{\bar{G}},$$

where $\bar{G} \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is the image of G modulo N . Observe that the modular curve X_G and its function field do not depend on the initial choice of N .

Every (open) subgroup G' of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ that contains G satisfies $-I \in G'$ and $\det(G') = \hat{\mathbb{Z}}^\times$, and we have a morphism $X_G \rightarrow X_{G'}$. With $G' = \mathrm{GL}_2(\hat{\mathbb{Z}})$, we obtain a morphism $\pi_G: X_G \rightarrow X_{G'} = \mathbb{P}_{\mathbb{Q}}^1$ to the j -line that agrees with $\pi_{\bar{G}}$. The following is equivalent to Proposition 2.6.

Proposition 2.7. *Let E be an elliptic curve defined over \mathbb{Q} with $j_E \notin \{0, 1728\}$. Then $\rho_E(\mathrm{Gal}_{\mathbb{Q}})$ is conjugate in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ to a subgroup of G if and only if j_E belongs to $\pi_G(X_G(\mathbb{Q}))$. \square*

2.7. Complex conjugation. Fix a subgroup G of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ satisfying $-I \in G$ and $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$. For our curve X_G to have rational points, we need G to contain an element that “looks like” complex conjugation.

Lemma 2.8. *For any elliptic curve E/\mathbb{Q} and integer $N > 1$, the group $\rho_{E,N}(\mathrm{Gal}_{\mathbb{Q}})$ contains an element that is conjugate in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.*

Proof. This follows from Proposition 3.5 of [Zyw15b] (and its proof for the cases $j_E \in \{0, 1728\}$). \square

Note that $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ are conjugate to each other in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ if N is odd. If G does not contain an element that is conjugate in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, then $X_G(\mathbb{Q})$ must be empty since $X_G(\mathbb{R})$ is finite (by [Zyw15b, Proposition 3.5]), hence empty, since X_G is non-singular.

3. GROUP THEORETIC COMPUTATIONS

We define an *admissible group* to be an open subgroup G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ for which the following conditions hold:

- G has prime power level,
- $-I \in G$ and $\det(G) = \hat{\mathbb{Z}}^\times$,
- G contains an element that is conjugate in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

The condition $\det(G) = \hat{\mathbb{Z}}^\times$ is needed for Proposition 2.7 since we have $\det(\rho_E(\mathrm{Gal}_{\mathbb{Q}})) = \hat{\mathbb{Z}}^\times$. If we were interested in elliptic curve defined over other number fields, then we could loosen this restriction which could increase the base field of the modular curve X_G .

The condition $-I \in G$ is also needed in Proposition 2.7. For an elliptic curve E/\mathbb{Q} , there is a quadratic twist E'/\mathbb{Q} , which automatically has the same j -invariant as E , such that $-I \in \rho_{E'}(\mathrm{Gal}_{\mathbb{Q}})$.

The last condition on G is necessary in order for $X_G(\mathbb{Q})$ to be non-empty, as explained in §2.6.

Proposition 3.1. *Let G be an admissible group of genus 0. The set $X_G(\mathbb{Q})$ is infinite.*

Proof. We have $X_G(\mathbb{R}) \neq \emptyset$ by Proposition 3.5 of [Zyw15a]. For primes p not dividing its prime power level the modular curve X_G has good reduction at p and $X_G(\mathbb{Q}_p) \neq \emptyset$, since the reduction of X_G to \mathbb{F}_p necessarily has rational points that can be lifted to \mathbb{Q}_p via Hensel’s lemma. Thus X_G has rational points locally at all but at most one place of \mathbb{Q} . The product formula for Hilbert symbols and the Hasse-Minkowski Theorem then imply that X_G has a rational point and is thus isomorphic to \mathbb{P}^1 and has infinitely many rational points. \square

Remark 3.2. As shown by Proposition 3.1, our three criteria for admissibility rule out genus 0 curves with no rational points. There are ten groups G of 2-power level that satisfy our first two criteria but not the third; these give rise to the ten pointless conics X_G found in [RZB15]. There are three such groups of 3-power level, three of 5-power level, and none of higher prime-power level.

Fix an integer $g \geq 0$. In this section, we explain how to enumerate all admissible subgroups G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$, up to conjugacy, that have genus at most g . We shall apply these methods with $g = 1$ to verify Theorem 3.3 below, and to find explicit representatives of these conjugacy classes of groups; Magma [BCP97] scripts that perform this enumeration can be found at [SZ16].

Theorem 3.3.

- (i) *Up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, there are 220 admissible subgroups of genus 0.*
- (ii) *Up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, there are 250 admissible subgroups of genus 1.*

Remark 3.4. The 220 admissible subgroups G of genus 0, up to conjugacy, are precisely those given in Tables 1–3. More precisely, for each entry of the table, we have an integer N and a set of generators that generates the image in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ of an admissible group of level N and genus 0.

Remark 3.5. The 28 admissible subgroups G of genus 1 that have infinitely many rational points, up to conjugacy, are precisely those given in Table 4, of which 27 have level 16 and 1 has level 11. The levels arising among the remaining 222 are 7, 8, 9, 11, 16, 17, 19, 27, 32, and 49.

For a fixed admissible group G of level N , let Γ be the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices whose image modulo N lies in the image of G modulo N ; the level of Γ necessarily divides N , and Γ contains $-I$. By Lemma 2.4(ii) and Remark 2.5, the modular curve X_G has the same genus as Γ .

The basic idea of our computation is to reverse the process above; we start with a congruence subgroup Γ of genus at most g and prime power level, and then enumerate the possible groups G that could produce Γ .

Let S_g be the set of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of prime power level that contain $-I$ and have genus at most g . We know that the set S_g is finite from a theorem of Dennin [Den74]. When $g \leq 24$, and in particular, for $g = 1$, we can explicitly determine the elements of S_g from the tables of Cummins and Pauli [CP03], which are available online at [CP] (their methods can also be extended to larger g).

Let L_g be the set of primes that divide the level of some congruence subgroup $\Gamma \in S_g$. The set L_g is finite, since S_g is finite, and we have $L_1 = \{2, 3, 5, 7, 11, 13, 17, 19\}$. If G is an admissible group of genus at most g , then its level must be a power of a prime $\ell \in L_g$. For the rest of the section, we fix a prime $\ell \in L_g$. Since L_g is finite, it suffices to explain how to compute the admissible groups G with genus at most g whose level is a power of ℓ , and we need only consider levels strictly greater than 1 since $\mathrm{GL}_2(\hat{\mathbb{Z}})$ is the only admissible group of level 1.

Fix a prime power $N := \ell^n > 1$, and consider any congruence subgroup $\Gamma \in S_g$ whose level divides N . By enumerating subgroups of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ one can explicitly determine those subgroups G_N that satisfy the following conditions:

- (1) G_N has level N ,
- (2) $G_N \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is equal to the image of Γ modulo N ,
- (3) $\det(G_N) = (\mathbb{Z}/N\mathbb{Z})^\times$,
- (4) G_N contains an element that is conjugate in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

Let H be the image of Γ in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. The group $H = G_N \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is normal in G_N and hence G_N is a subgroup of the normalizer K of H in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. So rather than searching for G_N in K , we can work in the quotient K/H where the image of G_N is an abelian group isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$. Using Magma, we can efficiently enumerate all abelian subgroups A of K/H of order $\#(\mathbb{Z}/N\mathbb{Z})^\times$. For each such subgroup A we then test whether its inverse image G_N in K satisfies conditions (1)–(4) above.

Let G be the subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ consisting of those matrices whose image modulo N lies in a fixed group G_N satisfying the conditions (1)–(4). The group G is admissible of level N and has

genus at most g . Moreover, it is clear that every admissible group of level N and genus at most g arises in this manner.

Fix an integer $e \geq 1$. By applying the above method with $1 \leq n \leq e$, we obtain all admissible groups G of genus at most g and level dividing ℓ^e . Our algorithm proceeds by applying this procedure to increasing values of e . In order for it to terminate we need to know that there are only finitely many admissible groups G of ℓ -power level and genus at most g , and we need an explicit way to determine when we have reached an e that is large enough to guarantee that we have found them all. Proposition 3.6 below addresses both issues.

Proposition 3.6.

- (i) *There are only finitely many admissible groups G with genus at most g whose level is a power of ℓ .*
- (ii) *Take any integer $n \geq 2$ with $n \neq 2$ if $\ell = 2$. Define $N := \ell^n$. Suppose that there is no subgroup G_N of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ that satisfies conditions (1)–(4) for some $\Gamma \in S_g$ with level dividing N . Then any admissible group G of genus at most g with level a power of ℓ has level at most N .*

The remainder of this section is devoted to proving Proposition 3.6. We will need the following basic lemma.

Lemma 3.7. *Let ℓ be a prime and let G be an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_\ell)$. For each integer $m \geq 1$, let i_m be the index of the image of G in $\mathrm{GL}_2(\mathbb{Z}/\ell^m\mathbb{Z})$. If $i_{n+1} = i_n$ for an integer $n \geq 1$, with $n \neq 1$ if $\ell = 2$, then $[\mathrm{GL}_2(\mathbb{Z}_\ell) : G] = i_n$.*

Proof. Since G is an open subgroup, it suffices to prove $i_{m+1} = i_m$ for all $m \geq n$; we proceed by induction on m . The base case is given, so we assume $i_{m+1} = i_m$ for some $m \geq n$; we need to show that $i_{m+2} = i_{m+1}$. Let G_m denote the image of G in $\mathrm{GL}_2(\mathbb{Z}/\ell^m\mathbb{Z})$. Reduction modulo ℓ^m gives exact sequences related by inclusions

$$\begin{array}{ccccccc} 1 & \longrightarrow & K_{m+1} & \longrightarrow & \mathrm{GL}_2(\mathbb{Z}/\ell^{m+1}\mathbb{Z}) & \longrightarrow & \mathrm{GL}_2(\mathbb{Z}/\ell^m\mathbb{Z}) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & H_{m+1} & \longrightarrow & G_{m+1} & \longrightarrow & G_m & \longrightarrow & 1. \end{array}$$

The inductive hypothesis $i_{m+1} = i_m$ implies that the kernels H_{m+1} and K_{m+1} coincide; in particular, H_{m+1} is as large as possible (i.e., it has order ℓ^4). It thus suffices to show that the kernel H_{m+2} of the reduction map from G_{m+2} to G_{m+1} also has order ℓ^4 . We have $|H_{m+2}| \leq \ell^4$, so it suffices to give an injective map $H_{m+1} \rightarrow H_{m+2}$.

Let M be an element of G whose image in G_{m+1} lies in H_{m+1} ; then $M = I + \ell^m A$ for some $A \in \mathrm{M}_2(\mathbb{Z}_\ell)$. Since $m \geq 1$, with $m \geq 2$ if $\ell = 2$, we have

$$(1 + \ell^m A)^\ell = 1 + \binom{\ell}{1} \ell^m A + \binom{\ell}{2} \ell^{2m} A^2 + \dots \equiv 1 + \ell^{m+1} A \pmod{\ell^{m+2}}.$$

The ℓ -power map thus induces an injection $H_{m+1} \rightarrow H_{m+2}$. □

Remark 3.8. Lemma 3.7 holds more generally. One can replace $\mathrm{GL}_2(\mathbb{Z}_\ell)$ with the unit group of any (unital associative) \mathbb{Z}_ℓ -algebra \mathcal{A} that is torsion-free and finitely-generated as a \mathbb{Z}_ℓ -module (in the lemma, $\mathcal{A} = \mathrm{M}_2(\mathbb{Z}_\ell)$); the proof is exactly the same.

Proof of Proposition 3.6(i). Let \mathcal{G} be the set of admissible groups of genus at most g whose level is a power of ℓ . Note that if G' is a subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ containing some $G \in \mathcal{G}$, then $G' \in \mathcal{G}$. We wish to show that \mathcal{G} is finite.

We claim that any admissible group G has only finitely many maximal subgroups that are also admissible and whose level is a power of ℓ . It suffices to show that an open subgroup H of $\mathrm{GL}_2(\mathbb{Z}_\ell)$

has only finitely many open maximal subgroups. Let $\Phi(H)$ be the Frattini subgroup of H ; it is the intersection of the maximal closed proper subgroups of H . By the proposition in §10.5 of [Ser97], $\Phi(H)$ is an open subgroup of H . This proves the claim.

Now suppose that \mathcal{G} is infinite. The claim implies that \mathcal{G} contains an infinite descending chain $G_1 \supsetneq G_2 \supsetneq G_3 \supsetneq \cdots$ (let $G_1 = \mathrm{GL}_2(\hat{\mathbb{Z}}) \in \mathcal{G}$, let $G_2 \in \mathcal{G}$ be one of the finitely many maximal subgroups of G_1 in \mathcal{G} that has infinitely many subgroups in \mathcal{G} , and continue in this fashion). For each $i \geq 1$, let Γ_i be the congruence subgroup associated to G_i (i.e., Γ_i consists of the matrices in $\mathrm{SL}_2(\mathbb{Z})$ whose image modulo N lies in the image modulo N of G_i , where N is the level of G_i); then $\Gamma_i \in S_g$. Since $[\mathrm{GL}_2(\hat{\mathbb{Z}}) : G_i] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_i]$, we have inclusions $\Gamma_1 \supsetneq \Gamma_2 \supsetneq \Gamma_3 \supsetneq \cdots$. This contradicts the finiteness of S_g and the proposition follows. \square

Proof of Proposition 3.6(ii). Fix an integer $n \geq 1$ as in the statement of part (ii). Suppose there is an integer $m > n$ such that there is an admissible group G of level ℓ^m and genus at most g .

With $N := \ell^n$, let G_N be the image of G in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. The curve X_{G_N} has genus at most g since it is dominated by X_G . Therefore, conditions (2), (3) and (4) hold for some $\Gamma \in S_g$ with level dividing N . Our assumption on n implies that the level of G_N is a proper divisor of N . This implies that the index $i_n := [\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : G_N]$ agrees with $i_{n-1} := [\mathrm{GL}_2(\mathbb{Z}/\ell^{n-1}\mathbb{Z}) : G_{\ell^{n-1}}]$, where $G_{\ell^{n-1}}$ is the image of G in $\mathrm{GL}_2(\mathbb{Z}/\ell^{n-1}\mathbb{Z})$. Since $i_n = i_{n-1}$, Lemma 3.7 implies that $[\mathrm{GL}_2(\mathbb{Z}) : G] = i_{n-1}$. However, this means that G has level dividing ℓ^{n-1} which is impossible since, by assumption, G has level $\ell^m > \ell^{n-1}$. Therefore, no such admissible group G exists. \square

4. CONSTRUCTION OF HAUPTMODULS

Fix a congruence subgroup Γ of genus 0 and level N . The function field of X_Γ is then of the form $\mathbb{C}(h)$, where the function $h: X_\Gamma \rightarrow \mathbb{C} \cup \{\infty\}$ gives an isomorphism between X_Γ and the Riemann sphere; in particular, h has a unique (simple) pole.

We may choose h so that its unique pole is at the cusp ∞ ; we will call such an h a *hauptmodul* of Γ . Every hauptmodul of Γ is then of the form $ah + b$ for some complex numbers $a \neq 0$ and b . For example, the familiar modular j -invariant

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

is a hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$. If h is a hauptmodul for Γ , then we have an inclusion of function fields $\mathbb{C}(j) \subseteq \mathbb{C}(h)$ and hence $J(h) = j$ for a unique rational function $J(h) \in \mathbb{C}(t)$.

The main task of §4 is to describe how to find an *explicit* hauptmodul h of Γ in terms of Siegel functions when N is a prime power. Our h will have coefficients in K_N . In §4.4, we explain how to compute the rational function $J(t)$ corresponding to h .

4.1. Siegel functions. Take any pair $a = (a_1, a_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$. We define the *Siegel function* $g_a(\tau)$ to be the holomorphic function $\mathbb{H} \rightarrow \mathbb{C}^\times$ defined by the series

$$-q^{\frac{1}{2}B_2(a_1)} \cdot e(a_2(a_1 - 1)/2) \cdot (1 - e(a_2)q^{a_1}) \prod_{n=1}^{\infty} (1 - e(a_2)q^{n+a_1})(1 - e(-a_2)q^{n-a_1}),$$

where $e(z) = e^{2\pi iz}$ and $B_2(x) = x^2 - x + 1/6$.

Recall that the *Dedekind eta function* is the holomorphic function on \mathbb{H} given by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

For each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there is a unique 12-th root of unity $\varepsilon(\gamma) \in \mathbb{C}^\times$ such that

$$(4.1) \quad \eta(\gamma\tau)^2 = \varepsilon(\gamma)(c\tau + d)\eta(\tau)^2.$$

We can characterize the map $\varepsilon: \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{C}^\times$ by the property that it is a homomorphism satisfying $\varepsilon\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \zeta_{12}$ and $\varepsilon\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \zeta_4$, cf. [KL81, Ch. 3 §5]. Moreover, the kernel of ε is a congruence subgroup of level 12 and agrees with the commutator subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

The following lemma gives several key properties of Siegel functions.

Lemma 4.1. *For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $a \in \mathbb{Q}^2 - \mathbb{Z}^2$, and $b \in \mathbb{Z}^2$, the following hold:*

- (i) $g_{-a} = -g_a$,
- (ii) $g_{a+b} = (-1)^{b_1+b_2+b_1b_2} \cdot e((b_2a_1 - b_1a_2)/2) \cdot g_a$,
- (iii) $g_a|_\gamma = \varepsilon(\gamma) \cdot g_{a\gamma}$, where we view a as a row vector.

Proof. In [KL81, Ch. 2 §1], we see that $g_a(\tau) = \mathfrak{k}_a(\tau)\eta(\tau)^2$, where $\mathfrak{k}_a(\tau)$ is a Klein form (with $W = W_\tau$ in the notation of loc. cit.). Part (ii) follows directly from property K2 of §1 of [KL81, Ch. 2].

Take any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and let (c, d) be the last row of γ . From properties K0 and K1 of §1 of [KL81, Ch. 2], we find that

$$(4.2) \quad \mathfrak{k}_a(\gamma\tau) = (c\tau + d)^{-1} \mathfrak{k}_{a\gamma}(\tau)$$

From (4.1) and (4.2), we deduce that $g_a(\gamma\tau) = \varepsilon(\gamma) \cdot g_{a\gamma}(\tau)$ which proves part (iii). Finally, part (i) follows from part (iii) with $\gamma = -I$, since $\varepsilon(-I) = -1$. \square

For an integer $N > 1$, let \mathcal{A}_N be the set of pairs $(a_1, a_2) \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}^2$ that satisfy one of the following conditions:

- $0 < a_1 < 1/2$ and $0 \leq a_2 < 1$,
- $a_1 = 0$ and $0 < a_2 \leq 1/2$,
- $a_1 = 1/2$ and $0 \leq a_2 \leq 1/2$.

The set \mathcal{A}_N is chosen so that every non-zero coset of $(N^{-1}\mathbb{Z}^2)/\mathbb{Z}^2$ is represented by an element of the form a or $-a$ for a unique $a \in \mathcal{A}_N$. So for any $a \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}^2$, we can use parts (i) and (ii) of Lemma 4.1 to show that

$$g_a = \epsilon \cdot \zeta \cdot g_{a'}$$

for an explicit sign $\epsilon \in \{\pm 1\}$, N -th root of unity ζ , and pair $a' \in \mathcal{A}_N$.

4.2. Siegel orbits. Now fix a congruence subgroup Γ of level $N > 1$. For each $a \in \mathcal{A}_N$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, let $a * \gamma$ be the unique element of \mathcal{A}_N such that $a * \gamma$ or $-a * \gamma$ lies in the coset $a\gamma + \mathbb{Z}^2$. The map

$$\mathcal{A}_N \times \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathcal{A}_N, \quad (a, \gamma) \mapsto a * \gamma$$

then gives a right action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathcal{A}_N . In particular, this gives a right action of Γ on \mathcal{A}_N .

Fix a Γ -orbit \mathcal{O} of \mathcal{A}_N and define

$$g_{\mathcal{O}} := \prod_{a \in \mathcal{O}} g_a;$$

it is a holomorphic function $\mathbb{H} \rightarrow \mathbb{C}^\times$.

Lemma 4.2. *The function $g_{\mathcal{O}}^{12N}$ is a modular function for Γ . Every pole and zero of $g_{\mathcal{O}}^{12N}$ on X_Γ is a cusp.*

Proof. Take any $\gamma \in \Gamma$ and $a \in \mathcal{A}_N$. By Lemma 4.1(iii), we have $g_a^{12N}|_\gamma = g_{a\gamma}^{12N}$. We have $a\gamma = \epsilon \cdot (a * \gamma + b)$ for some $\epsilon \in \{\pm 1\}$ and $b \in \mathbb{Z}^2$. By parts (i) and (ii) of Lemma 4.1, we find that $g_a^{12N}|_\gamma = g_{a\gamma}^{12N}$ is equal to $g_{a * \gamma}^{12N}$. Therefore,

$$g_{\mathcal{O}}^{12N}|_\gamma = \prod_{a \in \mathcal{O}} g_a^{12N}|_\gamma = \prod_{a \in \mathcal{O}} g_{a * \gamma}^{12N} = g_{\mathcal{O}}^{12N},$$

where the last equality uses the fact that the map $\mathcal{O} \rightarrow \mathcal{O}$, $a \mapsto a * \gamma$ is a bijection (since \mathcal{O} is a Γ -orbit). The remaining statement about the poles and zeros of $g_{\mathcal{O}}^{12N}$ follows immediately since each g_a is holomorphic and non-zero on \mathbb{H} . \square

Let P_1, \dots, P_r be the cusps of X_Γ . Choose a representative $s_j \in \mathbb{Q} \cup \{\infty\}$ of each cusp P_j and a matrix $A_j \in \mathrm{SL}_2(\mathbb{Z})$ satisfying $A_j \cdot \infty = s_j$. Let w_j be the *width* of the cusp P_j ; it is the smallest positive integer b such that $A_j \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} A_j^{-1}$ is an element of Γ .

For a non-zero meromorphic function f of \mathbb{H} given by a q -expansion, we define $\mathrm{ord}_q(f)$ to be the smallest rational number m such that there is a non-zero term of the form q^m in the expansion of f . For each cusp P_j , define the map

$$v_{P_j}: \mathbb{C}(X_\Gamma)^\times \rightarrow \mathbb{Z}, \quad f \mapsto w_j \cdot \mathrm{ord}_q(f|_{A_j});$$

it is a surjective homomorphism and agrees with the valuation giving the order of vanishing of a function at P_j . We extend ord_q and v_{P_j} by setting $\mathrm{ord}_q(0) = +\infty$ and $v_{P_j}(0) = +\infty$.

We now give a computable expression for the divisor of $g_{\mathcal{O}}^{12N}$ on X_Γ .

Lemma 4.3. *With notation as above, we have*

$$\mathrm{div}(g_{\mathcal{O}}^{12N}) = \sum_{j=1}^r \left(6Nw_j \sum_{a \in \mathcal{O}} B_2(\langle (aA_j)_1 \rangle) \right) \cdot P_j,$$

where $B_2(x) = x^2 - x + 1/6$, $(aA_j)_1$ is the first coordinate of the row vector aA_j , and $\langle x \rangle$ denotes the positive fractional part of the real number x , chosen so $0 \leq \langle x \rangle < 1$ and $x - \langle x \rangle \in \mathbb{Z}$.

Proof. For any $a \in (N^{-1}\mathbb{Z}^2) - \mathbb{Z}^2$, we have $\mathrm{ord}_q(g_a) = \frac{1}{2} \cdot B_2(\langle a_1 \rangle)$, cf. [KL81, p. 31]. We have

$$v_{P_j}(g_{\mathcal{O}}^{12N}) = \sum_{a \in \mathcal{O}} v_{P_j}(g_a^{12N}) = \sum_{a \in \mathcal{O}} w_j \mathrm{ord}_q(g_a^{12N}|_{A_j}) = \sum_{a \in \mathcal{O}} w_j \mathrm{ord}_q(g_{aA_j}^{12N}),$$

where the last equality uses Lemma 4.1(iii). Therefore,

$$v_{P_j}(g_{\mathcal{O}}^{12N}) = \sum_{a \in \mathcal{O}} 12Nw_j \mathrm{ord}_q(g_{aA_j}) = 6Nw_j \sum_{a \in \mathcal{O}} B_2(\langle (aA_j)_1 \rangle).$$

Since the poles and zeros of $g_{\mathcal{O}}^{12N}$ are all cusps, we have $\mathrm{div}(g_{\mathcal{O}}^{12N}) = \sum_{i=1}^r v_{P_i}(g_{\mathcal{O}}^{12N}) \cdot P_i$, and the lemma follows immediately. \square

4.3. Constructing hauptmoduls of prime power level. Fix a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of *prime power* level $N > 1$ that has genus 0. Let P_1, \dots, P_r be the cusps of Γ ; we choose our cusps so that P_1 is the cusp at ∞ .

In this section, we explain how to construct an explicit hauptmodul of Γ whose q -expansion has coefficients in K_N . Moreover, our hauptmodul will be of the form

$$(4.3) \quad \sum_{i=1}^M \zeta_{2N^2}^{e_i} \prod_{a \in \mathcal{A}_N} g_a^{m_{a,i}}$$

with integers $m_{a,i}$ and e_i .

4.3.1. Case 1: multiple cusps. First assume that Γ has at least two cusps. We will use the following lemma to construct a hauptmodul for certain genus 0 congruence subgroups.

Let $\mathcal{O}_1, \dots, \mathcal{O}_n$ be the distinct Γ -orbits of \mathcal{A}_N . For each \mathcal{O}_i , define the divisor $D_i := \mathrm{div}(g_{\mathcal{O}_i}^{12N})$ on X_Γ . By Lemma 4.3, the divisors D_1, \dots, D_n are supported on $\{P_1, \dots, P_r\}$ and are straightforward to compute.

Lemma 4.4. *Suppose there is an n -tuple $m \in \mathbb{Z}^n$ such that*

$$\sum_{i=1}^n m_i D_i = -12N \cdot P_1 + 12N \cdot P_2$$

Let $0 \leq e < 2N^2$ be the integer satisfying $e \equiv \sum_{i=1}^n m_i \sum_{a \in \mathcal{O}_i} Na_2(N - Na_1) \pmod{2N^2}$. Then

$$h := \zeta_{2N^2}^e \prod_{i=1}^n g_{\mathcal{O}_i}^{m_i}$$

is a hauptmodul for Γ whose q -expansion has coefficients in K_N . On X_Γ , we have $\operatorname{div}(f) = -P_1 + P_2$.

Proof. Since X_Γ has genus 0, there is a meromorphic function f on X_Γ with $\operatorname{div}(f) = -P_1 + P_2$. Lemma 4.2 implies that f^{12N}/h^{12N} defines a function on X_Γ ; it has divisor

$$12N \operatorname{div}(f) - \sum_{i=1}^n m_i \operatorname{div}(g_{\mathcal{O}_i}^{12N}) = 12N(-P_1 + P_2) - \sum_{i=1}^n m_i D_i = 0,$$

where the last equality uses our assumption on m . Therefore, f^{12N}/h^{12N} is constant. Since f and h are meromorphic functions on the upper half-plane, we deduce that f/h is a (non-zero) constant. In particular, h is modular for Γ and $\operatorname{div}(h) = -P_1 + P_2$. The function h on X_Γ is a hauptmodul for Γ since its only pole is the simple pole at P_1 , i.e., the cusp at ∞ .

It remains to show that the coefficients of h lie in K_N . Take any $a \in \mathcal{A}_N$. From the series defining g_a , we find that it is equal to the root of unity $e(a_2(a_1 - 1)/2) = \zeta_{2N^2}^{Na_2(Na_1 - N)}$ times a Laurent series in $q^{1/(6N^2)}$ with coefficients in K_N . Set $e' := \sum_{i=1}^n m_i \sum_{a \in \mathcal{O}_i} Na_2(Na_1 - N)$. The coefficients of $\zeta_{2N^2}^{-e'} \prod_{i=1}^n g_{\mathcal{O}_i}^{m_i}$ thus all lie in K_N . The lemma follows since $e \equiv -e' \pmod{2N^2}$. \square

Using the Cummins-Pauli classification of genus 0 congruence subgroups [CP03], we have explicitly verified that the n -tuple m from Lemma 4.4 always exists. Using Lemma 4.3, the existence of m comes down to finding integral solutions to r linear equations with integer coefficients in n variables. Using Lemma 4.4, we can thus find an explicit hauptmodul for Γ of the form (4.3) with $M = 1$ (we have $m_{a,i} = m_i$ if $a \in \mathcal{O}_i$).

Remark 4.5. One can also abstractly prove the existence of the n -tuple m . If N is an odd prime power, then any modular function of level N whose zeros and poles are all cusps can be expressed as a constant times a product of Siegel functions g_a with $a \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}$; cf. Theorem 1.1(i) of Chapter 5 of [KL81].

If $N \geq 4$ is a power of 2, this can also be deduced from Theorem 1.1(i) of Chapter 5 of [KL81]. (One needs to be a little careful here since g_a has a different definition in [KL81, Ch. 4 §1] when $2a \in \mathbb{Z}$. For the alternate g_a from loc. cit. with $2a \in \mathbb{Z}$, one can express them as a constant times a product of Siegel functions $g_{a'}$ with $a' \in \mathcal{A}_4 \subseteq \mathcal{A}_N$.)

The case $N = 2$ can be handled directly. For example, one can show that

$$g_{(1/2,0)}^8 \cdot g_{(1/2,1/2)}^4 \quad \text{and} \quad g_{(1/2,0)}^{12} \cdot g_{(1/2,1/2)}^{12}$$

are hauptmoduls for $\Gamma(2)$ and $\Gamma_0(2)$, respectively (note that $\Gamma_{\text{ns}}(2)$ has a single cusp and does not fall into this case, it falls into case 2 below).

4.3.2. Case 2: a single cusp and $N \neq 11$. Now assume that X_Γ has a single cusp and that $N \neq 11$. There are no non-constant modular functions for Γ whose zeros and poles are only at the cusps of X_Γ . In particular, a hauptmodul of Γ will never be equal to a product of Siegel functions.

Using the Cummins-Pauli classification, we find that there is a congruence subgroup Γ' that is a proper normal subgroup of Γ , also of level N and containing $-I$, such that $X_{\Gamma'}$ has genus 0 and has exactly $[\Gamma : \Gamma']$ cusps. (This is where we use $N \neq 11$.)

Since $X_{\Gamma'}$ has multiple cusps, we know from §4.3.1 how to construct a hauptmodul h' of Γ' with coefficients in K_N that is of the form (4.3). Using that Γ' is normal in Γ , we find that $h'|_\gamma$ is

modular for Γ' for all $\gamma \in \Gamma$ and the function depends only on the coset $\Gamma' \cdot \gamma$. Define

$$h := \sum_{\gamma \in \Gamma' \backslash \Gamma} h'|_{\gamma};$$

it is a modular function for Γ . Since X_{Γ} has only one cusp and $X_{\Gamma'}$ has $[\Gamma : \Gamma']$ cusps, we deduce that the modular functions $\{h'|_{\gamma}\}_{\gamma \in \Gamma' \backslash \Gamma}$ on $X_{\Gamma'}$ each have their unique (simple) pole at different cusps. This implies that h has a simple pole at the unique cusp of X_{Γ} and is holomorphic elsewhere. Therefore, h is a hauptmodul for Γ .

Since h' is modular for $\Gamma(N)$ and has coefficients in K_N , so does $h'|_{\gamma}$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$; see Proposition 2.2. Therefore, the coefficients of h lie in K_N .

Finally, it remains to show that h is of the form (4.3). It suffices to show that $h'|_{\gamma}$ is of the form (4.3) for a fixed $\gamma \in \Gamma$. We know that h' is equal to some product $\zeta_{2N^2}^e \prod_{a \in \mathcal{A}_N} g_a^{m_a}$, so $h|_{\gamma} = \varepsilon(\gamma)^b \zeta_{2N^2}^e \prod_{a \in \mathcal{A}_N} g_{a\gamma}^{m_a}$ with $b := \sum_{a \in \mathcal{A}_N} m_a$ by Lemma 4.1(iii). Recall that for each $a \in \mathcal{A}_N$, there is a unique $a * \gamma \in \mathcal{A}_N$ such that $a\gamma$ lies in the same coset of $(N^{-1}\mathbb{Z}^2)/\mathbb{Z}^2$ as $a * \gamma$ or $-a * \gamma$. From Lemma 4.1(i) and (ii), the functions $g_{a\gamma}^{m_a}$ and $g_{a*\gamma}^{m_a}$ agree up to a multiplication by some computable root of unity $-\zeta_N^e$. Therefore, $h|_{\gamma}$ is equal to $\varepsilon(\gamma)^b$ times a function of the form (4.3) with $M = 1$.

It remains only to show that $\varepsilon(\gamma)^b$ is a power of a $2N^2$ -th root of unity. In [KL81, Ch. 3 §5], Kubert and Lang give a necessary and sufficient condition for the product $\prod_{a \in \mathcal{A}_N} g_a^{m_a}$ to be modular for $\Gamma(N)$; these conditions hold since h' is modular for $\Gamma' \supseteq \Gamma(N)$. If N is a power of a prime $\ell \geq 5$, then Theorem 5.2 of Chapter 3 of [KL81] implies that $b \equiv 0 \pmod{12}$ and hence $\varepsilon(\gamma)^b = 1$. If N is a power of 3, then Theorem 5.3 of Chapter 3 of [KL81] implies that $b \equiv 0 \pmod{4}$ and hence $\varepsilon(\gamma)^b$ is a power of ζ_3 . If N is a power of 2, then Theorem 5.3 of Chapter 3 of [KL81] implies that $b \equiv 0 \pmod{3}$ and hence $\varepsilon(\gamma)^b$ is a power of ζ_4 . Therefore, $\varepsilon(\gamma)^b$ is indeed a power of a $2N^2$ -th root of unity.

4.3.3. *Case 3: $N = 11$.* The remaining case is when X_{Γ} has a single cusp and $N = 11$. We include this case only for completeness; we will not need it for our application.

Define the function

$$f(\tau) := \prod_{(a_1, a_2) \in B} g_{(a_1/11, a_2/11)}(\tau),$$

where

$$B := \left\{ \begin{array}{l} (0, 1), (0, 2), (0, 3), (1, 0), (1, 2), (1, 5), (1, 7), (2, 1), (2, 2), (2, 4), (2, 5), (2, 6), \\ (2, 7), (2, 8), (2, 9), (2, 10), (3, 0), (3, 2), (3, 4), (3, 5), (3, 6), (3, 8), (3, 10), (4, 0), \\ (4, 1), (4, 2), (4, 4), (4, 5), (4, 6), (5, 1), (5, 4), (5, 5), (5, 6), (5, 7), (5, 8), (5, 9) \end{array} \right\}.$$

One can verify that

$$\sum_{(a_1, a_2) \in B} a_1^2 \equiv \sum_{(a_1, a_2) \in B} a_2^2 \equiv \sum_{(a_1, a_2) \in B} a_1 a_2 \equiv 0 \pmod{11}$$

and that $|B| = 36 \equiv 0 \pmod{12}$. Theorem 5.2 of [KL81, Ch. 3 §5] implies that f is a modular function for $\Gamma(11)$. Using $\sum_{(a_1, a_2) \in B} a_2/11 \cdot (a_1/11 - 1)/2 = -60/11$ and the q -expansion of Siegel functions from §4.1, we find that all the coefficients of f lie in K_{11} . Therefore, $f \in \mathcal{F}_{11}$.

Using that $\Gamma(11)$ is normal in Γ , we find that $f|_{\gamma}$ is modular for $\Gamma(11)$ for all $\gamma \in \Gamma$ and the function depends only on the coset $\Gamma(11) \cdot \gamma$. Define

$$h := \sum_{\gamma \in \Gamma(11) \backslash \Gamma} f|_{\gamma};$$

it is a modular function for Γ . That h is of the form (4.3) follows as in the previous case.

We claim that h is a hauptmodul for Γ . From our description of h in terms of Siegel functions, we find that h has no poles except possibly at the unique cusp (at ∞). From Cummins and Pauli [CP03], there is a unique genus 0 congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level 11 up to conjugacy in $\mathrm{GL}_2(\mathbb{Z})$ (the one labeled 11A⁰). We have computed all the possible Γ and shown that h has a simple pole at ∞ , and is therefore a hauptmodul.

Remark 4.6. The set B comes from §5.3 of [CLY04]. In [CLY04], methods are given to compute hauptmoduls for genus 0 congruence subgroups (unfortunately, their hauptmodul tables are no longer available). They use “generalized Dedekind eta functions” which are essentially Siegel functions.

4.4. Rational function $J(t)$. For a hauptmodul h of Γ , there is a unique function $J(t) \in \mathbb{C}(t)$ such that $J(h) = j$; it has degree $d := [\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma]$.

Let us briefly explain how to compute $J(t)$ assuming that one can compute sufficiently many terms of the expansion of f . Let $K \subseteq \mathbb{C}$ be a field containing all the coefficients of h . Consider the equation

$$(4.4) \quad (a_d h^d + \dots + a_1 h + a_0) - j \cdot (b_d h^d + \dots + b_1 h + b_0) = 0$$

with unknowns $a_i, b_i \in K$, where $d := [\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma]$. Computing the q -expansion coefficients of the left-hand side of (4.4) yields a system of homogeneous linear equations in the unknowns a_i and b_i . The existence and uniqueness of J ensure that the solutions $(a_1, \dots, a_d, b_1, \dots, b_d) \in K^{2d}$ form a one-dimensional subspace. By computing sufficiently many coefficients of (4.4) one can find a non-zero solution $(a_1, \dots, a_d, b_1, \dots, b_d) \in K^{2d}$, unique up to scaling by K^\times , and

$$J(t) = (a_d t^d + \dots + a_1 t + a_0) / (b_d t^d + \dots + b_1 t + b_0) \in K(t)$$

is then the unique rational function for which $J(h) = j$. Note that if the hauptmodul h is constructed as in the previous section then we will have $J(t) \in K_N(t)$, where N is the level of Γ .

5. MODULAR CURVES OF GENUS 0

Fix the following:

- An integer $N > 1$ that is a prime power.
- A subgroup G of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ satisfying $-I \in G$ and $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$.
- A rational function $J(t) \in \mathbb{Q}(t)$.

In this section, we explain how to determine if the function field of X_G is of the form $\mathbb{Q}(f)$ for some modular function $f \in \mathcal{F}_N$ satisfying $J(f) = j$. We will use this to verify the entries of Tables 1–3.

If such an f exists, then X_G is isomorphic to $\mathbb{P}_{\mathbb{Q}}^1$ and the morphism $\pi_G: X_G \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ is given by the relation $j = J(f)$ in their function fields. We may assume the necessary condition that $[\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : G] = \deg \pi_G$ agrees with the degree of $J(t)$.

Remark 5.1. In §8 we explain how the $J(t)$ listed in Tables 1–3, were actually found, which involves the use of a Monte Carlo algorithm and assumes the Generalized Riemann Hypothesis (GRH). The purpose of this section is to explain how we can unconditionally verify a given $J(t)$, regardless of how it was found.

5.1. Construction of possible f . Let Γ be the congruence subgroup consisting of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ for which γ^t modulo N lies in G (equivalently, in $G \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$). By Lemma 2.4(ii), we may assume that Γ has genus 0 since otherwise X_G has positive genus and its function field cannot be of the form $\mathbb{Q}(f)$.

The group Γ acts on the right on the field \mathcal{F}_N ; let \mathcal{F}_N^Γ be subfield fixed by this action. By Lemma 2.4(i), we have $K_N(X_G) = \mathcal{F}_N^\Gamma$.

In §4.3, we described how to compute an explicit hauptmodul h for Γ such that coefficients of its q -expansion all lie in $K_{N'} \subseteq K_N$, where the level N' of Γ divides N . Therefore, we have

$$K_N(X_G) = \mathcal{F}_N^\Gamma = K_N(h).$$

Moreover, we can express h in terms of Siegel functions and hence we can compute as many of its coefficients as we desire. In §4.4, we described how to compute the unique rational function $J'(t) \in K_N(t)$ for which $j = J'(h)$. The degree of $J'(t)$ agrees with $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = [\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : G]$, thus $J(t)$ and $J'(t)$ have the same degree.

Remark 5.2. The rational function $J'(t)$ gives a map to the j -line from X_Γ , which is defined over $K_N = \mathbb{Q}(\zeta_N)$, while the rational function $J(t)$ gives a map to the j -line from X_G , which is defined over \mathbb{Q} . We use $J'(t)$ in our procedure to verify $J(t)$, but note that $J'(t)$ does not determine $J(t)$; in general there will be multiple non-conjugate subgroups G corresponding to Γ and a different rational function $J(t)$ for each of the corresponding X_G (in total we have 220 modular curves X_G of genus 0 corresponding to 73 modular curves X_Γ).

Lemma 5.3. *The modular functions $f \in K_N(X_G)$ that satisfy $K_N(X_G) = K_N(f)$ and $J(f) = j$ are precisely those of the form $\psi(h)$, where $\psi(t) \in K_N(t)$ is a degree 1 function satisfying $J'(t) = J(\psi(t))$.*

Proof. First take any $\psi(t) \in K_N(t)$ of degree 1 satisfying $J'(t) = J(\psi(t))$. Define $f := \psi(h)$. We have $K_N(f) = K_N(h) = K_N(X_G)$, since ψ has an inverse, and $J(f) = J(\psi(h)) = J'(h) = j$.

Now suppose that $K_N(X_G) = K_N(f)$ for some $f \in K_N(X_G)$ satisfying $J(f) = j$. Since $K_N(f) = K_N(X_G) = K_N(h)$, we have $f = \psi(h)$ for a unique $\psi(t) \in K_N(t)$ of degree 1. We then have $j = J(f) = J(\psi(h))$ and therefore $J'(t) = J(\psi(t))$, since $J'(t)$ is the unique element of $K_N(t)$ that satisfies $J'(h) = j$. \square

5.2. Finding possible f . Define Ψ to be the set of $\psi(t) \in K_N(t)$ of degree 1 for which $J'(t) = J(\psi(t))$; these ψ arise in Lemma 5.3. We now explain how to compute Ψ .

Choose three distinct elements $\beta_1, \beta_2, \beta_3 \in K_N \cup \{\infty\}$. For $1 \leq i \leq 3$, define the set

$$R_i := \{\alpha \in K_N \cup \{\infty\} : J'(\beta_i) = J(\alpha) \text{ and } \mathrm{ord}_{\beta_i}(J') = \mathrm{ord}_\alpha(J)\},$$

where $\mathrm{ord}_{\beta_i}(J')$ is the order of vanishing of $J'(t)$ at $t = \beta_i$. Let R be the set of triples $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in R_1 \times R_2 \times R_3$ such that α_1, α_2 and α_3 are distinct. Let $\psi_\alpha \in K_N(t)$ be the *unique* rational function of degree 1 such that $\psi_\alpha(\beta_i) = \alpha_i$ for all $1 \leq i \leq 3$.

Take any $\psi \in \Psi$. We have $J'(\beta_i) = J(\psi(\beta_i))$ and $\mathrm{ord}_{\beta_i}(J') = \mathrm{ord}_{\psi(\beta_i)}(J)$ for each $1 \leq i \leq 3$. Therefore, $\psi(\beta_i) \in R_i$ for each $1 \leq i \leq 3$ and hence $\psi = \psi_\alpha$ for some $\alpha \in R$. So we have

$$\Psi = \{\psi_\alpha : \alpha \in R, J'(t) = J(\psi(t))\}.$$

Since R is finite, this gives us a way to compute the (finite) set Ψ .

By Lemma 5.3, the set

$$\{\psi(h) : \psi \in \Psi\}$$

is the set of modular functions $f \in K_N(X_G)$ that satisfy $K_N(X_G) = K_N(f)$ and $J(f) = j$.

5.3. Checking each f . Let f be one of the finite number of functions that satisfy $K_N(X_G) = K_N(f)$ and $J(f) = j$. We just saw how to compute all such f ; they are of the form $\psi(h)$ for a degree 1 function $\psi(t) \in K_N(t)$ and a modular function h satisfying $K_N(X_G) = K_N(h)$ that is expressed in terms of Siegel functions. Recall from §2.4 that each $A \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on \mathcal{F}_N via the isomorphism $\theta_N : \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \xrightarrow{\sim} \mathrm{Gal}(\mathcal{F}_N/\mathbb{Q}(j))$ of Proposition 2.2, and for $f \in \mathcal{F}_N$ we use $A_*(f) := \theta_N(A)(f)$ to denote this action.

Lemma 5.4.

- (i) We have $\mathbb{Q}(X_G) = \mathbb{Q}(f)$ if and only if $f \in \mathbb{Q}(X_G)$.
- (ii) For a matrix $A \in G$, we have $A_*(f) = f$ if and only if $\text{ord}_q(A_*(f) - f) > 2w/N'$, where w is the width of the cusp ∞ of X_Γ and N' is the level of Γ .

Proof. We first prove part (i); only one implication needs proof. Suppose that $f \in \mathbb{Q}(X_G)$. Then $\mathbb{Q}(f) \subseteq \mathbb{Q}(X_G)$ and it suffices to show that these two fields have the same degree over $\mathbb{Q}(j)$. This is true since we have been assuming that $\deg \pi_G$ is equal to the degree of $J(t)$.

We now prove part (ii); only one implication needs proof. Suppose that $\text{ord}_q(A_*(f) - f) > 2w/N'$. As meromorphic functions on X_Γ , f and $A_*(f)$ have a unique (simple) pole since h has this property and ψ has degree 1. Therefore, the function $A_*(f) - f$ on X_Γ is zero or has at most two poles (and hence at most two zeros). Our assumption $\text{ord}_q(A_*(f) - f) > 2w/N'$ implies that $A_*(f) - f$ has a zero of order 3 at the cusp ∞ and thus $A_*(f) - f = 0$. \square

By Lemma 5.4(i), we have $\mathbb{Q}(X_G) = \mathbb{Q}(f)$ if and only if $A_*(f) = f$ for all $A \in G$ in a set of generators of G ; it suffices to consider $A \in G$ for which $\det(A)$ generate $(\mathbb{Z}/N\mathbb{Z})^\times$ since h and hence f is fixed by $G \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. It remains to describe how to determine whether $A_*(f)$ is equal to f . By Lemma 5.4(ii), it suffices to compute enough terms of the q -expansion of $A_*(f) - f$ to determine whether $\text{ord}_q(A_*(f) - f) > 2w/N'$ holds.

Finally, let us briefly explain how to compute terms in the q -expansion of $A_*(f) - f$. Let d be an odd integer congruent to $\det(A)$ modulo N . Choose a matrix $\gamma \in \text{SL}_2(\mathbb{Z})$ so that $A^t \equiv \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \gamma \pmod{N}$. We thus have

$$(5.1) \quad A_*(f) - f = \sigma_d(f)|_\gamma - f = \sigma_d(\psi)(\sigma_d(h)|_\gamma) - \psi(h),$$

where $\sigma_d(\psi)$ is the rational function with σ_d applies to the coefficients of its numerator and denominator. Our hauptmodul h is of the form $\sum_{i=1}^M \zeta_{2N^2}^{e_i} \prod_{a \in \mathcal{A}_N} g_a^{m_{a,i}}$ for certain integers e_i and $m_{a,i}$, so

$$\sigma_d(h)|_\gamma = \sum_{i=1}^M \zeta_{2N^2}^{e_i d} \prod_{a \in \mathcal{A}_N} (\sigma_d(g_a)|_\gamma)^{m_{a,i}}.$$

From the series expansion of g_a , one easily checks that $\sigma_d(g_{(a_1, a_2)}) = g_{(a_1, da_2)}$. From Lemma 4.1(iii), we have $\sigma_d(g_a)|_\gamma = \varepsilon(\gamma)g_{(a_1, da_2)\gamma}$ and hence

$$\sigma_d(h)|_\gamma = \sum_{i=1}^M \zeta_{2N^2}^{e_i d} \cdot \prod_{a \in \mathcal{A}_N} \varepsilon(\gamma)^{m_{a,i}} \cdot \prod_{a \in \mathcal{A}_N} g_{(a_1, da_2)\gamma}^{m_{a,i}}.$$

Thus by computing enough terms in the q -expansion of the functions $\{g_a\}_{a \in \mathcal{A}_N}$, we are able to compute the q -expansion of h and $\sigma_d(h)|_\gamma$ to as many terms as we desire. This allows us to compute terms in the q -expansion of $A_*(f) - f$ via (5.1).

Remark 5.5. Suppose that X_Γ has at least 3 cusps. We then have $A_*(f) = f$ if and only if $A_*(f)$ and f take the same the value at any three of the cusps (as in the proof of Lemma 5.4 this implies that $A_*(f) - f$ has at least three zeros and hence is the zero function). In the case of at least 3 cusps, our hauptmodul h was given as a constant times a product of Siegel functions; so its value at the cusp ∞ is determined by the first term of the q -expansion of h . The value at any other cusp c can be determined by the first term of the q -expansion of $h|_\gamma$ with $\gamma \in \text{SL}_2(\mathbb{Z})$ satisfying $\gamma\infty = c$. This approach is quicker since fewer terms of the q -expansions are required.

5.4. Verifying the entries of our tables. We now explain how to verify the validity of our genus 0 tables. Magma scripts that perform these verifications can be found at [SZ16].

Each row of Tables 1–3 gives a set of generators of a subgroup G of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ that satisfies $-I \in G$ and $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$ for a prime power N . We may assume that $N > 1$. By composing rational maps, we obtain a corresponding rational function $J(t) \in \mathbb{Q}(t)$.

Using the earlier parts of §5, we can construct a modular function $f \in \mathcal{F}_N$ such that $\mathbb{Q}(X_G) = \mathbb{Q}(f)$ and $J(f) = j$. So X_G is isomorphic to $\mathbb{P}_\mathbb{Q}^1$ and the morphism $\pi_G: X_G \rightarrow \mathbb{P}_\mathbb{Q}^1$ is given by the relation $j = J(f)$ in their function fields. (We also note that there is no harm in replacing G by a conjugate group; this is useful because one can reuse the hauptmodul computations for different groups in the tables).

Fix a group $G \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ as above, and a modular function $f \in \mathcal{F}_N$ satisfying $\mathbb{Q}(X_G) = \mathbb{Q}(f)$ and $J(f) = j$.

Now fix another group $G' \subseteq \mathrm{GL}_2(\mathbb{Z}/N'\mathbb{Z})$ from our table so that N divides N' and the image of G' in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is conjugate to a subgroup of G . In the above computations, we have constructed a modular function f' satisfying $\mathbb{Q}(X_{G'}) = \mathbb{Q}(f')$ and $J'(f') = j$ for a rational function $J'(t) \in \mathbb{Q}(t)$ also arising from the tables.

Take any subgroup $\tilde{G} \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ conjugate to G' whose image modulo N lies in G . Choose any $A \in \mathrm{GL}_2(\mathbb{Z}/N'\mathbb{Z})$ for which $\tilde{G} := AG'A^{-1}$ and define $\tilde{f} := A_*(f')$. We have an inclusion of fields

$$\mathbb{Q}(\tilde{f}) = \mathbb{Q}(X_{\tilde{G}}) \supseteq \mathbb{Q}(X_G) = \mathbb{Q}(f).$$

The extension $\mathbb{Q}(\tilde{f})/\mathbb{Q}(f)$ has degree $i := [\mathrm{GL}_2(\mathbb{Z}/N'\mathbb{Z}):G']/[\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}):G]$. Therefore, $\varphi(\tilde{f}) = f$ for a unique $\varphi(t) \in \mathbb{Q}(t)$ of degree i . We can compute $\varphi(t)$ using the method from §4.4; the coefficients of f and \tilde{f} can be computed as in §5.3.

The rational function φ is not unique, it depends on the choices of \tilde{G} , f , f' , and A . However, any other rational function occurring would be of the form $\psi'(\varphi(\psi(t)))$, where $\psi, \psi' \in \mathbb{Q}(t)$ are degree 1 functions satisfying $J(\psi(t)) = J(t)$ and $J'(\psi'(t)) = J'(t)$. Note that all the possible ψ and ψ' can be computed as in §5.2 (with $J = J'$). We have checked that the rational function relating G and G' in our tables, when given, is indeed of the form $\psi'(\varphi(\psi(t)))$.

6. MODULAR CURVES OF GENUS 1

We now consider the open subgroups G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ with genus 1 and prime power level $N = \ell^e$ that satisfy $-I \in G$ and $\det(G) = \hat{\mathbb{Z}}^\times$. We are interested in describing those G for which $X_G(\mathbb{Q})$ is infinite. There is no harm in replacing G by a conjugate. So by Theorem 3.3(ii), there are 250 cases that need to be checked.

Let J_G be the Jacobian of the curve X_G . Using the methods of [Zyw15b], we can compute the rank of $J_G(\mathbb{Q})$. From [DR73, §IV], we find that the curve X_G has good reduction at all primes $p \nmid N = \ell^e$. Therefore, J_G is an elliptic curve defined over \mathbb{Q} whose conductor is a power of ℓ . The primes ℓ that arise are small enough to ensure that J_G is isomorphic to one of the elliptic curves in Cremona's tables [Cre14]; this gives a finite number of candidates for J_G up to isogeny.

For each prime $p \nmid 6\ell$, we can compute $\#J_G(\mathbb{F}_p) = \#X_G(\mathbb{F}_p)$ from the modular interpretation of X_G , cf. [Zyw15b, §3.6] for details. In particular, we can compute $\#J_G(\mathbb{F}_p)$ directly from the group G without computing a model for X_G (or its reduction modulo p). By computing several values of $\#J_G(\mathbb{F}_p)$ with $p \neq \ell$, we can quickly distinguish the isogeny class of J_G among the finite set of candidates. We then compute the rank of $J_G(\mathbb{Q})$, which we note is an isogeny invariant.

Running this procedure on each of the 250 genus 1 groups G given by Theorem 3.3, we find that $J_G(\mathbb{Q})$ has rank 0 for 222 groups and $J_G(\mathbb{Q})$ has positive rank for 28 groups; a Magma script that performs this computation can be found in [SZ16]. We need only consider the 28 groups G for which $J_G(\mathbb{Q})$ has positive rank, since $X_G(\mathbb{Q})$ is finite if $J_G(\mathbb{Q})$ has rank 0.

Now let G be one of the 28 groups for which $J_G(\mathbb{Q})$ has positive rank; they are precisely the 28 genus 1 groups in Theorem 1.1 and can be found in Table 4 in Appendix A. For each of these groups G , if $X_G(\mathbb{Q})$ is nonempty then it must be infinite, since the Abel-Jacobi map then gives a

bijection from $X_G(\mathbb{Q})$ to $J_G(\mathbb{Q})$. We initially verified that $X_G(\mathbb{Q})$ is nonempty by finding an elliptic curve E/\mathbb{Q} with $\rho_E(\text{Gal}_{\mathbb{Q}}) \subseteq G$ using an extension of the algorithm in [Sut16].

For each of these 28 groups G , a model for X_G and the morphism π_G can already be found in the literature (and are equivalent to the ones we give in Appendix A). For the 27 groups G of level 16 these curves and morphisms were constructed by Rouse and Zureick-Brown in [RZB15]; the models and morphisms we give in Table 4 for these groups are slightly different (we constructed them by taking fiber products of our genus 0 curves), but we have verified that they are isomorphic (note that their groups are transposed relative to ours). The remaining group G has level 11 and its image in $\text{GL}_2(\mathbb{Z}/11\mathbb{Z})$ is the normalizer of a non-split Cartan subgroup. An explicit model for $X_G = X_{\text{ns}}^+(11)$ and the morphism to the j -line can be found in [Hal98]; these are reproduced in Appendix A.

7. PROOF OF THEOREM 1.4

If $\ell \leq 13$, then the set \mathcal{J}_ℓ is finite by [Zyw15b, Proposition 4.8]. If $\ell > 13$, this follows from [Zyw15b, Proposition 4.9]; note that ρ_{E,ℓ^∞} is surjective if and only if $\rho_{E,\ell}$ is surjective, since $\ell \geq 5$, by [Ser89, IV Lemma 3]. This proves (i).

For a group G from Theorem 1.1, define the set

$$\mathcal{S}_G := \bigcup_{G'} \pi_{G',G}(X_{G'}(\mathbb{Q})),$$

where G' varies over the proper subgroups of G that are conjugate to one of groups in Theorem 1.1 of ℓ -power level and $\pi_{G',G}: X_{G'} \rightarrow X_G$ is the natural morphism induced by the inclusion $G' \subseteq G$. Note that this is a finite union.

Suppose first that G has genus 0. Then $X_G \simeq \mathbb{P}_{\mathbb{Q}}^1$ and \mathcal{S}_G is a *thin* subset of $X_G(\mathbb{Q})$, in the language of [Ser97, §9]. It follows that $X_G(\mathbb{Q}) - \mathcal{S}_G$ is Zariski dense in $X_G \simeq \mathbb{P}_{\mathbb{Q}}^1$, hence infinite, by Hilbert irreducibility.

Suppose that G has genus 1. If G does not have level 16 and index 24, then there are no proper subgroups G' of G that are conjugate to a group from Theorem 1.1, and therefore \mathcal{S}_G is empty and $X_G(\mathbb{Q}) - \mathcal{S}_G$ is infinite.

Now suppose that G has genus 1, level 16, and index 24. There are 7 such G , labeled

$$16C^1-16c, 16C^1-16d, 16B^1-16a, 16B^1-16c, 16D^1-16d, 8D^1-16b, 8D^1-16c$$

and explicitly described in Table 4 of Appendix A. Each of these G contains either 2 or 4 index 2 subgroups G' that are conjugate to one of the groups in Theorem 1.1. In every case we have $\mathcal{S}_G = X_G(\mathbb{Q})$, so that $X_G(\mathbb{Q}) - \mathcal{S}_G$ is empty; see Example 6.11 and Remark 6.3 in [RZB15].

Let E/\mathbb{Q} be an elliptic curve with $j_E \notin \mathcal{J}_\ell$. The group $\pm\rho_{E,\ell^\infty}(\text{Gal}_{\mathbb{Q}})$ is conjugate in $\text{GL}_2(\mathbb{Z}_\ell)$ to the ℓ -adic projection of a unique group G from Theorem 1.1 with ℓ -power level. Using Proposition 2.6, we can also characterize G as the unique group from Theorem 1.1 with ℓ -power level such that $j_E \in \pi_G(X_G(\mathbb{Q}) - \mathcal{S}_G)$. Parts (ii) and (iii) follow by noting that $\pi_G(X_G(\mathbb{Q}) - \mathcal{S}_G)$ is empty when G has genus 1, level 16, and index 24, and it is infinite otherwise.

8. HOW THE $J(t)$ WERE FOUND

Let G be one of the genus 0 subgroups of $\text{GL}_2(\hat{\mathbb{Z}})$ from Theorem 1.1; they are listed in Tables 1–3 in Appendix A and were determined using the algorithm described in §3. For each G , we also have a rational function $J(t) \in \mathbb{Q}(t)$ such that the function field of X_G is of the form $\mathbb{Q}(f)$ and $j = J(f)$, where j is the modular j -invariant; the verification of this property is described in §5.

In this section, we explain how we found $J(t)$; note that the method we used to verify the correctness of $J(t)$ does not depend on how it was found! None of our theorems depend on the

techniques described in this section. All that matters is that they eventually produced functions $J(t)$ whose correctness we could verify using the procedure described in §5.4.

We used an extension of the algorithm in [Sut16] to search for elliptic curves E/\mathbb{Q} for which $\rho_E(\text{Gal}_{\mathbb{Q}})$ is conjugate to a subgroup of G . This was initially done by simply checking elliptic curves in Cremona's tables [Cre14] and the LMFDB [LMF15] (but see Remark 8.1 below). After enough searching, we find elliptic curves E_1, E_2, E_3 defined over \mathbb{Q} with distinct j -invariants j_1, j_2, j_3 for which we believe that $\rho_{E_i}(\text{Gal}_{\mathbb{Q}})$ is conjugate in $\text{GL}_2(\hat{\mathbb{Z}})$ to a subgroup of G ; in particular, we expect that $j_1, j_2, j_3 \in \pi_G(X_G(\mathbb{Q}))$. We ran the Monte Carlo algorithm in [Sut16] using parameters that ensure the error probability is less than 2^{-100} , under the GRH.

Now suppose that j_1, j_2, j_3 are indeed elements of $\pi_G(X_G(\mathbb{Q}))$. The curve X_G has genus 0 and rational points, so it is isomorphic to $\mathbb{P}_{\mathbb{Q}}^1$. We can choose an isomorphism $X_G \simeq \mathbb{P}_{\mathbb{Q}}^1$ such that there are points $P_1, P_2, P_3 \in X_G(\mathbb{Q})$ satisfying $\pi_G(P_i) = j_i$ which map to $0, 1, \infty$, respectively. There is thus a rational function $J(t) \in \mathbb{Q}(t)$ such that $J(0) = j_1, J(1) = j_2, J(\infty) = j_3$ and such that $\mathbb{Q}(X_G) = \mathbb{Q}(f)$ for a modular function f satisfying $J(f) = j$; the function f is obtained by composing our isomorphism $\mathbb{P}_{\mathbb{Q}}^1 \simeq X_G$ with π_G .

We can now find all such potential J . As explained in §5, we can construct a modular function $h \in \mathcal{F}_N$ and a rational function $J'(t) \in K_N(t)$ such that $K_N(X_G) = K_N(h)$ and $j = J'(h)$, where N is the level of G . We thus have

$$J(t) = J'(\psi(t))$$

for some degree 1 function $\psi(t) \in K_N(t)$ satisfying $\psi(0) \in R_1, \psi(1) \in R_2$ and $\psi(\infty) \in R_3$, where $R_i := \{\alpha \in K_N \cup \{\infty\} : J'(\alpha) = j_i\}$. Since the sets R_i are finite and disjoint, there are only finitely many $\psi(t) \in \mathbb{Q}(t)$ of degree 1 satisfying $\psi(0) \in R_1, \psi(1) \in R_2, \psi(\infty) \in R_3$. For each such $\psi(t)$, we check whether $J'(\psi(t))$ lies in $\mathbb{Q}(t)$.

Consider any ψ as above for which $J'(\psi(t)) \in \mathbb{Q}(t)$. Set $J(t) := J'(\psi(t))$ and $f := \psi^{-1}(h) \in K_N(X_G)$. We have $J(f) = J'(h) = j$. The field $\mathbb{Q}(f)$ is thus the function field of a modular curve $X_{G'}$, where G' is an open subgroup of $\text{GL}_2(\hat{\mathbb{Z}})$ of level N satisfying $\det(G') = \hat{\mathbb{Z}}^\times$ and $-I \in G'$; it consists of matrices whose reduction modulo N fix f . We can then check whether G is equal to G' . Since $[\text{GL}_2(\hat{\mathbb{Z}}) : G] = \deg \pi_G = \deg J = [\text{GL}_2(\hat{\mathbb{Z}}) : G']$, it suffices to determine whether G is a subgroup of G' ; equivalently, whether G fixes f . A method for determining whether f is fixed by G is described in §5.3.

We will eventually find a ψ for which we have $G = G'$ (provided that our initial j -invariants j_i are valid). This then proves that $\mathbb{Q}(X_G) = \mathbb{Q}(f)$ for some f satisfying $J(f) = j$, where $J(t) := J'(\psi(t)) \in \mathbb{Q}(t)$.

Note this rational function $J(t)$ is not unique since $J(\varphi(t))$ would also work for any $\varphi(t) \in \mathbb{Q}(t)$ of degree 1. Using similar reasoning, it is easy to determine if two $J_1, J_2 \in \mathbb{Q}(t)$ satisfy $J_2(t) = J_2(\varphi(t))$ for some degree 1 function $\varphi \in \mathbb{Q}(t)$. We have chosen our rational functions so that they are relatively compact when written down.

Remark 8.1. Having run this procedure to obtain functions $J(t)$ for each of the groups G where we were able to find suitable E_1, E_2, E_3 in Cremona's tables, we then address the remaining groups G by picking a group G' that contains a subgroup conjugate to G for which we already know a function $J'(t) \in \mathbb{Q}(t)$; such a G' existed for every G not addressed in our initial search of Cremona's tables. Using the function $J'(t)$ we can quickly obtain a large list of elliptic curves E for which $\rho_E(\text{Gal}_{\mathbb{Q}})$ is a subgroup of G' . By running the algorithm in [Sut16] on several thousand (or even millions) of these curves we are eventually able to find E_1, E_2, E_3 with distinct j -invariants for which it is highly probable that $\rho_{E_i}(\text{Gal}_{\mathbb{Q}})$ is actually conjugate to a subgroup of the smaller group G contained in G' . We then proceed as above to compute the function $J(t)$ for G .

APPENDIX A. TABLES

This section includes tables that list data for all the groups G , up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, from Theorem 1.1. The genus 0 groups are given in Tables 1, 2 and 3. The genus 1 groups are given in Table 4.

We now describe how to read Tables 1–4. Each row corresponds to a unique group G from Theorem 1.1 up to conjugacy; it is given a unique label of the form MZ^g-Nz , where M , N and g are integers and Z and z are letters. The integers N and g are the level and genus of G , respectively. Let Γ be the congruence subgroup consisting of matrices in $\mathrm{SL}_2(\mathbb{Z})$ whose image modulo N lies in the image of G modulo N . The integer g is also the genus of the Riemann surface obtained by taking the quotient of the complex upper-half plane by the action of Γ . The integer M is the level of Γ and Z is an uppercase letter that distinguishes Γ up to conjugacy in $\mathrm{GL}_2(\mathbb{Z})$; the prefix MZ^g for Γ matches the label used by Cummins and Pauli [CP03]. The letter z is chosen so that the label MZ^g-Nz distinguishes G up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$. In some of the tables, we also number the rows.

For each row of these tables, there is a positive integer N and a set of matrices S in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$; the corresponding open subgroup G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ consists of the matrices $A \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ whose image in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ lies in the subgroup generated by S . The integer i in each row is the index $[\mathrm{GL}_2(\hat{\mathbb{Z}}) : G]$.

For each row of a table, we also have a list of minimal supergroups up to conjugacy (given by labels or numberings of other rows); more precisely, the groups G' satisfying $G \subsetneq G' \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$, up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, for which there are no subgroups strictly between G and G' .

If G has genus 0, then X_G is isomorphic to $\mathbb{P}_{\mathbb{Q}}^1$; its function field is of the form $\mathbb{Q}(t)$. If G has genus 1, then the curve X_G is isomorphic to the elliptic curve in Table 5 given by the “curve” column of Table 4; its function field is of the form $\mathbb{Q}(x, y)$, where x and y are the given Weierstrass coordinates.

Now for simplicity, suppose that G has genus 0. The function F listed in the “map” column describes the morphism $X_G \rightarrow X_{G'}$ for one of the corresponding minimal supergroups G' of G (or the first minimal supergroup listed if only one map is given). More precisely, after appropriately conjugating G' satisfying $G \subsetneq G'$, the morphism $X_G \rightarrow X_{G'}$ is given by the rational function $F(t) \in \mathbb{Q}(t)$, i.e., the curves X_G and $X_{G'}$ have function fields $\mathbb{Q}(t)$ and $\mathbb{Q}(u)$, respectively, and are related by the equation $u = F(t)$. By composing these rational maps down to $X_{\mathrm{GL}_2(\hat{\mathbb{Z}})} = \mathbb{P}_{\mathbb{Q}}^1$ (i.e., to the group labeled 1A⁰-1a), we obtain from G a rational function

$$J(t) \in \mathbb{Q}(t);$$

in §5, we showed that $J(t)$ describes the morphism π_G from X_G to the j -line. If $J'(t) \in \mathbb{Q}(t)$ is the rational function arising from G' in the same manner, then we will have $J(t) = J'(F(t))$. The function $J(t)$ is independent of any choice of supergroups.

Similar remarks and conventions hold when G has genus 1.

#	label	i	N	generators	map	sup
0	1A ⁰ -1a	1	1			
1	3A ⁰ -3a	3	3	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	t^3	0
2	3B ⁰ -3a	4	3	$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$	$(t+3)^3(t+27)/t$	0
3	3C ⁰ -3a	6	3	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$(t-9)(t+3)/t$	1
4	3D ⁰ -3a	12	3	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$729/(t^3-27)$	2
					$-27(t-3)/(t^2+3t+9)$	3
5	9A ⁰ -9a	9	9	$\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	t^3+9t-6	1
6	9B ⁰ -9a	12	9	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$t(t^2+9t+27)$	2
7	9C ⁰ -9a	12	9	$\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	t^3	2
8	9D ⁰ -9a	18	9	$\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$-27/t^3$	3
					$(t^2-3)/t$	5
9	9E ⁰ -9a	18	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 0 & 5 \end{pmatrix}$	$-9(t^3+3t^2-9t-3)/(8t^3)$	1
10	9F ⁰ -9a	27	9	$\begin{pmatrix} 0 & 2 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 0 & 5 \end{pmatrix}$	$\frac{3^7(t^2-1)^3(t^6+3t^5+6t^4+t^3-3t^2+12t+16)^3(2t^3+3t^2-3t-5)}{(t^3-3t-1)^9}$	0
11	9G ⁰ -9a	27	9	$\begin{pmatrix} 0 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 3 \\ 0 & 4 \end{pmatrix}$	$\frac{(t^3-9t-12)(9-3t^3)(5t^3+18t^2+18t+3)}{(t^3+3t^2-3)^3}$	1
12	9H ⁰ -9a	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$	$3(t^3+9)/t^3$	4
					$3t/(2t^2-3t+6)$	9
13	9H ⁰ -9b	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$3(t^3+9t^2-9t-9)/(t^3-9t^2-9t+9)$	4
14	9H ⁰ -9c	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 0 & 5 \end{pmatrix}$	$-6(t^3-9t)/(t^3+9t^2-9t-9)$	4
					$-(t^2+3)/(t^2+8t+3)$	10
15	9I ⁰ -9a	36	9	$\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$	$-6(t^3-9t)/(t^3-3t^2-9t+3)$	6
16	9I ⁰ -9b	36	9	$\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 3 & 5 \end{pmatrix}$	$-3(t^3+9t^2-9t-9)/(t^3+3t^2-9t-3)$	6
17	9I ⁰ -9c	36	9	$\begin{pmatrix} 2 & 2 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$	$(t^3-6t^2+3t+1)/(t^2-t)$	6
18	9J ⁰ -9a	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 3 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$	$(t^3-3t+1)/(t^2-t)$	7
19	9J ⁰ -9b	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 3 & 8 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$-18(t^2-1)/(t^3-3t^2-9t+3)$	7
20	9J ⁰ -9c	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$	$3(t^3+3t^2-9t-3)/(t^3-3t^2-9t+3)$	7
21	27A ⁰ -27a	36	27	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 9 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$	t^3	6

TABLE 1. Genus zero groups of 3-power level

label	i	N	generators	map	supergroup
1A ⁰ -1a	1	1			
5A ⁰ -5a	5	5	$(\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 \\ 2 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix})$	$t^3(t^2 + 5t + 40)$	1A ⁰ -1a
5B ⁰ -5a	6	5	$(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$(t^2 + 10t + 5)^3 / t$	1A ⁰ -1a
5C ⁰ -5a	10	5	$(\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 \\ 2 & 0 \end{smallmatrix}), (\begin{smallmatrix} 2 & 2 \\ 2 & 1 \end{smallmatrix})$	$8000t^3(t+1)(t^2-5t+10)^3 / (t^2-5)^5$	1A ⁰ -1a
5D ⁰ -5a	12	5	$(\begin{smallmatrix} 4 & 0 \\ 1 & 4 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$125t / (t^2 - 11t - 1)$	5B ⁰ -5a
5D ⁰ -5b	12	5	$(\begin{smallmatrix} 4 & 0 \\ 1 & 4 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})$	$(t^2 - 11t - 1) / t$	5B ⁰ -5a
5E ⁰ -5a	15	5	$(\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 2 & 3 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 2 & 2 \end{smallmatrix})$	$(t+5)(t^2-5) / (t^2+5t+5)$	5A ⁰ -5a
5G ⁰ -5a	30	5	$(\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}), (\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix})$	$125 / (t(t^4 + 5t^3 + 15t^2 + 25t + 25))$ $(t^2 + 5) / t$	5E ⁰ -5a 5B ⁰ -5a
5G ⁰ -5b	30	5	$(\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}), (\begin{smallmatrix} 2 & 1 \\ 3 & 3 \end{smallmatrix})$	$-t(t^2 + 5t + 10) / (t^3 + 5t^2 + 10t + 10)$ $-5(t^2 + 4t + 5) / (t^2 + 5t + 5)$	5C ⁰ -5a 5E ⁰ -5a
5H ⁰ -5a	60	5	$(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})$	$-1 / t^5$ $\frac{-(t^4 - 2t^3 + 4t^2 - 3t + 1)}{t(t^4 + 3t^3 + 4t^2 + 2t + 1)}$ $5t / (t^2 - t - 1)$	5D ⁰ -5a 5D ⁰ -5b 5G ⁰ -5a
25A ⁰ -25a	30	25	$(\begin{smallmatrix} 2 & 2 \\ 0 & 13 \end{smallmatrix}), (\begin{smallmatrix} 4 & 1 \\ 3 & 1 \end{smallmatrix}), (\begin{smallmatrix} 2 & 3 \\ 0 & 6 \end{smallmatrix})$	$(t-1)(t^4 + t^3 + 6t^2 + 6t + 11)$	5B ⁰ -5a
25B ⁰ -25a	60	25	$(\begin{smallmatrix} 9 & 10 \\ 0 & 14 \end{smallmatrix}), (\begin{smallmatrix} 0 & 7 \\ 7 & 2 \end{smallmatrix}), (\begin{smallmatrix} 2 & 8 \\ 0 & 1 \end{smallmatrix})$	$-t^5$	5D ⁰ -5b
25B ⁰ -25b	60	25	$(\begin{smallmatrix} 9 & 10 \\ 0 & 14 \end{smallmatrix}), (\begin{smallmatrix} 0 & 7 \\ 7 & 2 \end{smallmatrix}), (\begin{smallmatrix} 4 & 1 \\ 0 & 7 \end{smallmatrix})$	$(1 - t^2) / t$ $\frac{-(t^4 - 2t^3 + 4t^2 - 3t + 1)}{t(t^4 + 3t^3 + 4t^2 + 2t + 1)}$ $(t^2 + 4t - 1) / (t^2 - t - 1)$	25A ⁰ -25a 5D ⁰ -5a 25A ⁰ -25a
7B ⁰ -7a	8	7	$(\begin{smallmatrix} 2 & 0 \\ 0 & 4 \end{smallmatrix}), (\begin{smallmatrix} 3 & 0 \\ 1 & 5 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix})$	$(t^2 + 5t + 1)^3(t^2 + 13t + 49) / t$	1A ⁰ -1a
7D ⁰ -7a	21	7	$(\begin{smallmatrix} 2 & 3 \\ 0 & 3 \end{smallmatrix}), (\begin{smallmatrix} 2 & 4 \\ 4 & 5 \end{smallmatrix}), (\begin{smallmatrix} 3 & 1 \\ 0 & 4 \end{smallmatrix})$	$\frac{(2t-1)^3(t^2-t+2)^3(2t^2+5t+4)^3(5t^2+2t-4)^3}{(t^3+2t^2-t-1)^7}$	1A ⁰ -1a
7E ⁰ -7a	24	7	$(\begin{smallmatrix} 6 & 0 \\ 1 & 6 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix})$	$49(t^2 - t) / (t^3 - 8t^2 + 5t + 1)$	7B ⁰ -7a
7E ⁰ -7b	24	7	$(\begin{smallmatrix} 6 & 0 \\ 1 & 6 \end{smallmatrix}), (\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix})$	$(t^3 - 8t^2 + 5t + 1) / (t^2 - t)$	7B ⁰ -7a
7E ⁰ -7c	24	7	$(\begin{smallmatrix} 6 & 0 \\ 1 & 6 \end{smallmatrix}), (\begin{smallmatrix} 3 & 0 \\ 0 & 4 \end{smallmatrix})$	$-7(t^3 - 2t^2 - t + 1) / (t^3 - t^2 - 2t + 1)$	7B ⁰ -7a
7F ⁰ -7a	28	7	$(\begin{smallmatrix} 3 & 1 \\ 4 & 4 \end{smallmatrix}), (\begin{smallmatrix} 4 & 4 \\ 1 & 3 \end{smallmatrix}), (\begin{smallmatrix} 3 & 4 \\ 0 & 4 \end{smallmatrix})$	$\frac{t(t+1)^3(t^2-5t+1)^3(t^2-5t+8)^3(t^4-5t^3+8t^2-7t+7)^3}{(t^3-4t^2+3t+1)^7}$	1A ⁰ -1a
13A ⁰ -13a	14	13	$(\begin{smallmatrix} 2 & 0 \\ 0 & 7 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$(t^2 + 5t + 13)(t^4 + 7t^3 + 20t^2 + 19t + 1)^3 / t$	1A ⁰ -1a
13B ⁰ -13a	28	13	$(\begin{smallmatrix} 3 & 0 \\ 0 & 9 \end{smallmatrix}), (\begin{smallmatrix} 4 & 0 \\ 1 & 10 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$13t / (t^2 - 3t - 1)$	13A ⁰ -13a
13B ⁰ -13b	28	13	$(\begin{smallmatrix} 3 & 0 \\ 0 & 9 \end{smallmatrix}), (\begin{smallmatrix} 4 & 0 \\ 1 & 10 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})$	$(t^2 - 3t - 1) / t$	13A ⁰ -13a
13C ⁰ -13a	42	13	$(\begin{smallmatrix} 5 & 0 \\ 0 & 8 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$13(t^2 - t) / (t^3 - 4t^2 + t + 1)$	13A ⁰ -13a
13C ⁰ -13b	42	13	$(\begin{smallmatrix} 5 & 0 \\ 0 & 8 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})$	$(t^3 - 4t^2 + t + 1) / (t^2 - t)$	13A ⁰ -13a
13C ⁰ -13c	42	13	$(\begin{smallmatrix} 5 & 0 \\ 0 & 8 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix})$	$-(5t^3 - 7t^2 - 8t + 5) / (t^3 - 4t^2 + t + 1)$	13A ⁰ -13a

TABLE 2. Genus zero groups of ℓ -power level for primes $\ell > 3$

label	Weierstrass model
121b1	$y^2 + y = x^3 - x^2 - 7x + 10$
128a1	$y^2 = x^3 + x^2 + x + 1$
128a2	$y^2 = x^3 + x^2 - 9x + 7$
256a1	$y^2 = x^3 + x^2 - 3x + 1$
256a2	$y^2 = x^3 + x^2 - 13x - 21$
256b1	$y^2 = x^3 - 2x$
256b2	$y^2 = x^3 + 8x$

TABLE 5. Some elliptic curves

REFERENCES

- [BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). [↑3](#)
- [CLY04] Kok Seng Chua, Mong Lung Lang, and Yifan Yang, *On Rademacher’s conjecture: congruence subgroups of genus zero of the modular group*, J. Algebra **277** (2004), no. 1, 408–428. [↑4.6](#)
- [Cre14] John E. Cremona, *Elliptic curve data*, 2014. <http://homepages.warwick.ac.uk/staff/J.E.Cremona/ftp/data/>. [↑6, 8](#)
- [CP03] C. J. Cummins and S. Pauli, *Congruence subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ of genus less than or equal to 24*, Experiment. Math. **12** (2003), no. 2, 243–255. MR2016709 (2004i:11037) [↑1.2, 3, 4.3.1, 4.3.3, A](#)
- [CP] ———, *Database of congruence subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ of genus less than or equal to 24*. available at <http://www.uncg.edu/mat/faculty/pauli/congruence/>. [↑3](#)
- [DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 143–316. Lecture Notes in Math., Vol. 349. MR0337993 (49 #2762) [↑1, 6](#)
- [Den74] Jr. Dennis Joseph B., *The genus of subfields of $K(p^n)$* , Illinois J. Math. **18** (1974), 246–264. MR0342466 (49 #7212) [↑3](#)
- [Fal83] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366, DOI 10.1007/BF01388432 (German). MR718935 (85g:11026a) [↑1](#)
- [Hal98] Emmanuel Halberstadt, *Sur la courbe modulaire $X_{nd\acute{e}p}(11)$* , Experiment. Math. **7** (1998), no. 2, 163–174 (French, with English and French summaries). MR1677158 (99m:11062) [↑6](#)
- [KL81] Daniel S. Kubert and Serge Lang, *Modular units*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 244, Springer-Verlag, New York-Berlin, 1981. MR648603 (84h:12009) [↑2.4, 4.1, 4.1, 4.2, 4.5, 4.3.2, 4.3.3](#)
- [LMF15] LMFDB Collaboration, *The L-functions and Modular Forms Database*, 2015. <http://www.lmfdb.org>. [↑8](#)
- [NS03] Nikolay Nikolov and Dan Segal, *Finite index subgroups in profinite groups*, C. R. Math. Acad. Sci. Paris **337** (2003), no. 5, 303–308 (English, with English and French summaries). MR2016979 [↑1](#)
- [RZB15] Jeremy Rouse and David Zureick-Brown, *Elliptic curves over \mathbb{Q} and 2-adic images of Galois*, Research in Number Theory **1** (2015). [↑1.3, 3.2, 6, 7](#)
- [Ser72] Jean-Pierre Serre, *Propriétés galoisiennes des points d’ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), no. 4, 259–331. MR0387283 (52 #8126) [↑1](#)
- [Ser81] ———, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. **54** (1981), 323–401. MR644559 (83k:12011) [↑1](#)
- [Ser89] ———, *Abelian l -adic representations and elliptic curves*, Second Ed., Advanced Book Classics, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989. With the collaboration of Willem Kuyk and John Labute. MR1043865 (91b:11071) [↑7](#)
- [Ser97] ———, *Lectures on the Mordell-Weil theorem*, Third, Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1997. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt, With a foreword by Brown and Serre. MR1757192 (2000m:11049) [↑3, 7](#)
- [Shi94] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original; Kanô Memorial Lectures, 1. MR1291394 (95e:11048) [↑2.2](#)

- [Sut16] Andrew V. Sutherland, *Computing images of Galois representations attached to elliptic curves*, Forum of Mathematics, Sigma **4** (2016), 79 pages. ↑[1.1](#), [6](#), [8](#), [8.1](#)
- [SZ16] Andrew V. Sutherland and David Zywina, *Magma scripts associated to “Modular curves of prime-power level with infinitely many rational points”*, 2016. available at <http://math.mit.edu/~drew/SZ16>. ↑[1.1](#), [3](#), [5.4](#), [6](#)
- [Zyw15a] David Zywina, *On the possible images of the mod ℓ representations associated to elliptic curves over \mathbb{Q}* , 2015. arXiv:1508.07660v1 [math.NT]. ↑[1.1](#), [2.5](#), [3](#)
- [Zyw15b] ———, *Possible indices for the Galois image of elliptic curves over \mathbb{Q}* , 2015. arXiv:1508.07663v1 [math.NT]. ↑[ii](#), [2.7](#), [6](#), [7](#)

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, USA
E-mail address: drew@math.mit.edu
URL: <http://math.mit.edu/~drew>

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853, USA
E-mail address: zywina@math.cornell.edu
URL: <http://www.math.cornell.edu/~zywina>