Trace and extension results for a class of domains with self-similar boundary

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An example of ramified domain
Motivations

Human lungs (E.R. Weibel)
Lena river delta
Outline

1 Presentation
   • The self-similar boundary
   • The class of ramified domains

2 Trace and extension results
   • General results
   • Trace theorems in the critical case
   • Extension theorem in the critical case

3 Comparison of the notions of trace
A class of self-similar sets

Consider an iterated function system \((f_1, f_2)\), where
- \(f_1\) and \(f_2\) have ratio \(a < 1\),
- \(f_1\) and \(f_2\) have opposite angles \(\pm \theta\) \((0 \leq \theta < \frac{\pi}{2})\).

Denote \(\Gamma\) the invariant set associated with \((f_1, f_2)\) :
\[
\Gamma = f_1(\Gamma) \cup f_2(\Gamma).
\]
A class of self-similar sets

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An example of self-similar set \(\Gamma\)
A class of self-similar sets

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An example of self-similar set \(\Gamma\)

The critical ratio $a^*$

There exists a critical ratio $a^*$ depending only on the angle $\theta$ such that

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If $a \leq a^*$, Hausdorff dimension of $\Gamma$:

$$d := \dim_H(\Gamma) = -\frac{\log 2}{\log a}$$
Sobolev spaces on $\Gamma$

The self-similar set $\Gamma$ is endowed with its invariant measure, i.e. the only probability measure $\mu$ on $\Gamma$ satisfying

$$
\mu(B) = \frac{1}{2} \mu(f_1^{-1}(B)) + \frac{1}{2} \mu(f_2^{-1}(B))
$$

for every Borel set $B \subset \Gamma$. 
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**The spaces $W^{s,p}(\Gamma)$ (Jonsson, Wallin, 1984)**

For $0 < s < 1$ and $1 \leq p < \infty$, if $v \in L^p_\mu(\Gamma)$, then $v \in W^{s,p}(\Gamma)$ if and only if

$$\|v\|_{W^{s,p}(\Gamma)}^p := \iint_{|x-y|<1} \frac{|v(x) - v(y)|^p}{|x-y|^{d+ps}} \, d\mu(x) \, d\mu(y) < \infty.$$
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The class of ramified domains

Γ^0

First cell \( Y^0 \)
The class of ramified domains

\[ f_1(\Gamma^0) \quad f_2(\Gamma^0) \]

First cell \( Y^0 \)
The class of ramified domains

First cell $\Gamma^0$ $\gamma^0$ $f_1(\Gamma^0)$ $f_2(\Gamma^0)$
The class of ramified domains

First cell $Y^0$

Second iteration

$\Gamma^0$
The class of ramified domains

If \( \sigma = (\sigma_1, \ldots, \sigma_k) \in \{1, 2\}^k \), write

\[
f_\sigma := f_{\sigma_1} \circ \ldots \circ f_{\sigma_k}.
\]

\[
\Omega = \text{Interior} \left( \bigcup_{k \geq 0} \bigcup_{\sigma \in \{1, 2\}^k} f_\sigma(\overline{Y^0}) \right)
\]
The set $\Xi$ of multiple points of $\Gamma$

\[
a = a^*
\]

Case 1

$\theta \notin \{ \frac{\pi}{2k}, \ k \in \mathbb{N}^* \}$

$\Xi$ is countable

where $\Xi$ is the set of multiple points of $\Gamma$. 
The set $\Xi$ of multiple points of $\Gamma$

$a = a^*$

**Case 1**

$\theta \notin \left\{ \frac{\pi}{2k}, \ k \in \mathbb{N}^* \right\}$

$\Xi$ is countable

**Case 2**

$\theta = \frac{\pi}{2k}, \ k \in \mathbb{N}^*$

$\dim_H \Xi = \frac{d}{2}$

where $\Xi$ is the set of multiple points of $\Gamma$. 
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Questions

- **Trace:**
  - Notion of trace for functions in $W^{1,p}(\Omega)$ on $\Gamma$?
  - Sobolev regularity of the trace on $\Gamma$?
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- **Trace:**
  - Notion of trace for functions in $W^{1,p}(\Omega)$ on $\Gamma$?
  - Sobolev regularity of the trace on $\Gamma$?

- **Extension:**
  - For which values of $p$ is the domain $\Omega$ a $W^{1,p}$-extension domain, i.e. there exists a linear continuous extension operator

  $$W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^2)?$$

  If $\Omega$ is a $W^{1,p}$-extension domain for every $p \in [1, \infty]$, $\Omega$ is said to be a **Sobolev extension domain**.
Trace theorems

Theorem (Gagliardo, 1957)

If $\omega \subset \mathbb{R}^n$ is an open set with Lipschitz boundary and $1 < p < \infty$, one has the trace result:

$$W^{1,p}(\omega)_{|\partial \omega} = W^{1-\frac{1}{p},p}(\partial \omega).$$
Trace theorems

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**Theorem (A. Jonsson, H. Wallin, 1984)**

If $1 < p < \infty$, and $1 - \frac{2-d}{p} > 0$, then

$$W^{1,p}(\mathbb{R}^2)|_{\Gamma} = W^{1-\frac{2-d}{p},p}(\Gamma).$$
Trace and extension results

Trace theorems

**Theorem (Gagliardo, 1957)**

*If* \( \omega \subset \mathbb{R}^n \) *is an open set with Lipschitz boundary and* \( 1 < p < \infty \), *one has the trace result:*

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\]

**Theorem (A. Jonsson, H. Wallin, 1984)**

*If* \( 1 < p < \infty \), *and* \( 1 - 2 - \frac{d}{p} > 0 \), *then*

\[
W^{1,p}(\mathbb{R}^2) |_{\Gamma} = W^{1 - \frac{2-d}{p}, p}(\Gamma)
\]

**Sense of the trace:** *u* *is strictly defined at* \( x \in \mathbb{R}^2 \) *if the limit*

\[
\bar{u}(x) := \lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) \, dy
\]

exists. *Trace of* \( u \) *on* \( \Gamma \) : \( \bar{u}|_{\Gamma} \).
Extension theorems

Theorem (A.P. Calderón, E.M. Stein, 1970)

Every Lipschitz domain in $\mathbb{R}^n$ is a Sobolev extension domain.
Extension theorems

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*Every Lipschitz domain in $\mathbb{R}^n$ is a Sobolev extension domain.*

Jones domains (P.W. Jones, 1981):

A domain $\omega \subset \mathbb{R}^n$ is a **Jones domain** if there exist $\varepsilon, \delta > 0$ such that for every $x, y \in \omega$ satisfying $|x - y| < \delta$, there exists a rectifiable arc $\gamma \subset \omega$ joining $x$ to $y$ such that

- $L(\gamma) \leq \frac{1}{\varepsilon} |x - y|$ where $L(\gamma)$ = length of $\gamma$,
- $d(z, \partial \omega) \geq \varepsilon \min(|x - z|, |y - z|)$ for all $z \in \gamma$.

Theorem (P.W. Jones, 1981)

*Jones domains are Sobolev extension domains.*
The subcritical case \((a < a^*)\)

- **Extension**
  
  If \(\Omega\) is a ramified domain with \(a < a^*\), then \(\Omega\) is a Jones domain, so it has the \(p\)-extension property for all \(1 \leq p \leq \infty\).

- **Traces**
  
  Jones theorem combined with Jonsson and Wallin’s trace operator yields a trace operator:
  
  \[
  W^{1,p}(\Omega) \rightarrow W^{1 - \frac{2 - d}{p}, p}(\Gamma)
  \]
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The ramified domains with $a = a^*$

The case $a = a^*$

In this case, $\Omega$ cannot be a $W^{1,p}$-extension domain for $p > 2$. 
Haar wavelets on $\Gamma$:

\[
\begin{align*}
g_0 &= \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)}
\end{align*}
\]
Haar wavelets on $\Gamma$

Haar wavelets on $\Gamma$:

\[
\begin{align*}
g_0 &= 1_{f_1(\Gamma)} - 1_{f_2(\Gamma)} \\
g_\sigma|_{f_\sigma(\Gamma)} &= 2^k g_0 \circ f_\sigma^{-1} \quad \text{and} \quad g_\sigma|_{\Gamma \setminus f_\sigma(\Gamma)} = 0 \quad \text{for} \quad \sigma \in \{1, 2\}^k
\end{align*}
\]

Mother wavelet $g_0$
Haar wavelets on $\Gamma$

Haar wavelets on $\Gamma$:

\[
\begin{align*}
  g_0 &= \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)} \\
  g_\sigma|_{f_\sigma(\Gamma)} &= 2^{k/2} g_0 \circ f_\sigma^{-1} \quad \text{and} \quad g_\sigma|_{\Gamma \setminus f_\sigma(\Gamma)} = 0 \quad \text{for} \quad \sigma \in \{1, 2\}^k
\end{align*}
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Mother wavelet $g_0$

$g_\sigma$ for $\sigma = (1)$
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\[
\begin{align*}
  g_0 &= 1 f_1(\Gamma) - 1 f_2(\Gamma) \\
  g_\sigma|_{f_\sigma(\Gamma)} &= 2^{\frac{k}{2}} g_0 \circ f_\sigma^{-1} \text{ and } g_\sigma|_{\Gamma \setminus f_\sigma(\Gamma)} = 0 \text{ for } \sigma \in \{1, 2\}^k
\end{align*}
\]

Mother wavelet $g_0$

$g_\sigma$ for $\sigma = (1)$

Every function $v \in L^p_\mu(\Gamma)$, $1 \leq p < \infty$ can be expanded in the Haar wavelet basis $(g_\sigma)$:

\[
v = \langle v \rangle_\Gamma + \sum_{k \geq 0} \sum_{\sigma \in \{1, 2\}^k} \beta_\sigma g_\sigma.
\]
Construction of a self-similar trace operator

For \( u \in W^{1,p}(\Omega) \), define:

\[
\ell^0(u) \equiv \langle u \rangle_{\Gamma^0},
\]

where \( \langle u \rangle_{\Gamma^\sigma} = \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} u \).
Construction of a self-similar trace operator

\[ \ell^1(u) = \langle u \rangle_{f_1(\Gamma^0)} \mathbb{1}_{f_1(\Gamma)} + \langle u \rangle_{f_2(\Gamma^0)} \mathbb{1}_{f_2(\Gamma)} \]
Construction of a self-similar trace operator

\[ \ell^2(u) = \sum_{\sigma \in \{1,2\}^2} \langle u \rangle_{f_\sigma(\Gamma^0)} f_\sigma(\Gamma) \]
Construction of a self-similar trace operator

\[ \ell^2(u) = \sum_{\sigma \in \{1,2\}^2} \langle u \rangle_{f_\sigma(\Gamma^0)} \mathbb{1}_{f_\sigma(\Gamma)} \]

\[ \ell^n(u) = \sum_{\sigma \in \{1,2\}^n} \langle u \rangle_{f_\sigma(\Gamma^0)} \mathbb{1}_{f_\sigma(\Gamma)} \]
Construction of a self-similar trace operator

\[ \ell^2(u) = \sum_{\sigma \in \{1,2\}^2} \langle u \rangle f_{\sigma}(\Gamma^0) \mathbb{1}_{f_{\sigma}(\Gamma)} \]

\[ \ell^n(u) = \sum_{\sigma \in \{1,2\}^n} \langle u \rangle f_{\sigma}(\Gamma^0) \mathbb{1}_{f_{\sigma}(\Gamma)} \]

The sequence \((\ell^n)\) converges in \(L(\mathcal{W}^{1,p}(\Omega), L^p_{\mu}(\Gamma))\) to an operator \(\ell^\infty\).

Set \( p^* = \begin{cases} 2 & \text{in case 1} \\ 2 - \frac{d}{2} & \text{in case 2} \end{cases} = 2 - \dim_H \Xi \)

If \( a = a^* \), then
- if \( p < p^* \),
  \[ \ell^\infty(\mathcal{W}^{1,p}(\Omega)) = \mathcal{W}^{1, \frac{2-d}{p} \cdot p}(\Gamma) \]
- if \( p \geq p^* \),
  \[ \ell^\infty(\mathcal{W}^{1,p}(\Omega)) \subset \mathcal{W}^{s,p}(\Gamma) \]

for every \( s < \begin{cases} \frac{d}{p} & \text{in case 1} \\ \frac{d}{2p} & \text{in case 2} \end{cases} = \frac{d-\dim_H \Xi}{p} \)

\( p^* = 2 \)

\( p^* = 2 - \frac{d}{2} \)

Set $p^* = \begin{cases} 2 & \text{in case 1} \\ 2 - \frac{d}{2} & \text{in case 2} \end{cases}$

If $a = a^*$, then

- if $p < p^*$,
  
  $$\ell^\infty(\mathcal{W}^{1,p}(\Omega)) = \mathcal{W}^{1-\frac{2-d}{p},p}(\Gamma)$$

- if $p \geq p^*$,
  
  $$\ell^\infty(\mathcal{W}^{1,p}(\Omega)) \subset \mathcal{W}^{s,p}(\Gamma)$$

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In the case of a ramified domain with 4 similitudes and $\dim \Xi = \frac{\dim_H \Gamma}{4}$, the result holds.
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Extension theorem

In the case $a = a^*$,

- we know that a ramified domain $\Omega$ is not a $W^{1,p}$-extension domain for $p > 2$.
- the trace theorem suggests that $\Omega$ is not a $W^{1,p}$-extension domain for $p > p^*$.
- $p < p^*$?
Extension theorem

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- the trace theorem suggests that $\Omega$ is not a $W^{1,p}$-extension domain for $p > p^*$.
- $p < p^*$?

**Theorem (T.D., 2013)**

*If $\Omega$ is a critical ramified domain and $p^* = 2 - \dim \Xi$, then for all $p < p^*$, $\Omega$ is a $W^{1,p}$-extension domain*
Comparison of the notions of trace

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Comparison of the notions of trace

The following theorem justifies *a posteriori* the use of several notions of trace on $\Gamma$.


For $1 < p < \infty$, every function $u \in W^{1,p}(\Omega)$ is strictly defined $\mu$-almost everywhere on $\Gamma$, and

$$\overline{u}|_{\Gamma} = \ell^{\infty}(u)$$

$\mu$-almost everywhere on $\Gamma$.

The proof uses as a key ingredient the extension operator for $p < p^*$. In this case, the trace on $\Gamma$ does not depend on the extension operator: $(\mathcal{E} u)|_{\Gamma} =: \overline{u}|_{\Gamma}$.

**Corollary**

If $p > p^*$, $\Omega$ is not a $W^{1,p}$-extension domain.
Thank you for your attention!